# On the Cardinal Points in Plane Kinematics

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# ON THE CARDINAL POINTS IN PLANE KINEMATICS

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CONTENTS



In the elementary theory of geometrical optics all the non-aberration properties of the symmetrical optical system may be derived from the three pairs of cardinal points, due to Gauss, of which two pairs only are independent. And it is the case, in the general theory of plane kinematics, that there are certain points playing a somewhat similar role. For, associated with any relative co-planar motion of two planes, there are two enumerable sets of points, from the configuration of either of which may be derived all the properties of the relative path of any, and every, point, or series of points, fixed in either plane. The configuration of each of these sets of points is uniquely characteristic of the particular relative motion of the two planes, and conversely; and gives a very simple and compact synthesis of the whole realm of plane kinematics.

The purpose of the following investigation is to establish the existence of these 'cardinal points', as we may name them, in plane kinematics, and to examine some of their properties. In addition, various new curves and configurations are obtained, relating to the generation of the *Burmester points*, and similar points of higher orders; together with a generalization of certain kinematical results which have emerged, stage by stage, in the writings of Tschebycheff, Burmester, Müller and others.

## 1. Introduction

In a plane collineation there are three self-corresponding points, and, of such collineations, that one of more immediate importance in plane kinematics—for which lengths are conserved—has two of these points coincident with the absolute points. The existence of the third, and real, point implies that a plane may be transferred to any coplanar position by means of a pure rotation; and this has long been familiar. Indeed, it appears to have been known for a finite displacement before it was recognized for an infinitesimal displacement. Even earlier, de la Hire (1706) investigated various properties of roulettes, and to him also is due the introduction of the inflexion circle.

In the eighteenth century, Watts introduced a simple mechanism designed to generate approximate straight-line motion, from non-linear motion. And Tschebycheff, in a series of investigations extending over some thirty-four years (1854–1888), discussed the kinematical properties of this, and of allied, mechanisms. In particular, as associated with the three-bar mechanism, he discovered a certain property, afterwards generalized by Müller for all mechanisms, and capable of still further generalization, to the effect that, when contact of a certain order has been obtained, with a straight line, there are then three collinear points of the mechanism, the paths of two having complete contact with circles, as guided by the mechanism, and that of the third five-point contact with a third circle. The first of these four points coincides with the *Ball point*; for Ball found that, in quite general co-planar motion, there exists, at each instant, one point, and only one, the path of which has four-point contact with a straight line.

The writings of Burmester brought into prominence a certain cubic curve of importance in plane kinematics, and this curve has been the subject of numerous papers. There followed a long series of investigations, associated with the names of Burmester, Müller, Grubler, Rodenberg, Wittenbauer and others. These arose frequently from the study of particular mechanisms—usually the three-bar mechanism—and from them emerged results of a general kinematical nature; for example, the existence, in quite general coplanar motion, of four points—the *Burmester points*—the path of each of which has five-point contact with a corresponding circle. Further, if one of these points coincides with the Ball point then the remaining three are collinear; and it was this latter case, for a particular mechanism, which arose in the writings of Tschebycheff.

Towards the close of the above-mentioned series of papers emerged a certain chain of points associated with plane kinematics. For example, the centres of curvature of the envelopes of all straight lines fixed in a moving plane lie, at any given instant, upon a circle passing through the instantaneous centre, the opposite end of the diameter of which—the so-called *Rückkehrpol*—is a point of interest; as also is the *Wendepol*, the corresponding point in the dual motion. And each of these is the first of a chain of points. But some, at least, of these points appeared in different fashion (Léauté), as the centres of zero acceleration, and of zero hyper-acceleration, for uniform angular velocity of the moving plane. The points, however, are of more fundamental importance, in the general theory, than seems to appear here; save only, as far as I am aware, perhaps in some measure, in the last paper of the series, by Müller (1902).

In the elementary theory of geometrical optics all the non-aberration properties of the general symmetrical system are completely determined by the three pairs of cardinal points, introduced by Gauss, of which two pairs only are independent. And a main purpose of the present investigation is to establish the existence of certain *cardinal points* in plane kinematics, from knowledge of which, in somewhat similar fashion, may be determined completely all the properties of the path of any point, or series of points, of either of two planes in relative coplanar motion. The configuration of these cardinal points is characteristic of the relative motion, and conversely; and gives a very simple and compact synthesis of the whole realm of plane kinematics. In addition, various new curves and configurations are obtained, relating to the generation of the *Burmester points*, and similar points of higher orders, together with a generalization of kinematical results which have emerged, stage by stage, in the writings of Tschebycheff, Burmester, Müller and others.

There is a quite extensive, though somewhat scattered, literature dealing with plane kinematics, extending over a considerable period. Frequently, several finitely displaced positions of a plane are considered, and the corresponding geometrical properties are investigated; then, as a limiting case, the properties arising in consecutive infinesimal displacements are derived. But mention should also be made of a more recent treatment in quite different, and I believe quite novel, fashion by Blaschke (1938).

## 2. The cardinal points of the first set

2·1. We consider two planes $p$ and $\varpi$, subject to relative coplanar motion, or displacement, and we assume this relative motion to have one degree of freedom, specified by a parameter $\phi$, as in the usual mechanism, or linkage. We proceed to establish the existence of an enumerable set of points $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \ldots$, having the following characteristic: the relative path of an arbitrary point $P_\varpi$, fixed in the plane $\varpi$, has all its properties determined by its geometrical relationship with the set of points $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \ldots$. And, further, it is the case that the configuration of this set of points, for any value of the parameter $\phi$, determines the configurations of the set for all values of $\phi$. This configuration, for a single value of the parameter $\phi$, affords therefore a simple and complete synthesis of the generalized relative motion, or displacement, of the two planes $p$ and $\varpi$.

There is similarly a dual set of cardinal points $\mathscr{A}'_1, \mathscr{A}'_2, \ldots, \mathscr{A}'_n, \ldots$, arising from consideration of the relative path of an arbitrary point $P_p$, fixed in the plane $p$; and, clearly, the configuration of either set determines completely that of the other set. It is the case that $\mathscr{A}_1$ and $\mathscr{A}'_1$ coincide with the point commonly known as the instantaneous centre; but, in general, $\mathscr{A}_n$ and $\mathscr{A}'_n$, $n>1$, do not coincide. And, further, these points are, in general, fixed in neither plane. Their several loci, in the two planes, form a series of curves intimately related with the mechanism, or linkage, and, in particular, for $n=1$, we have the well-known centrodes, with their familiar rolling property.

Let $Oxy$ be rectangular axes fixed in a plane $p$, and $\Omega\xi\eta$ similar axes fixed in a superimposed plane $\varpi$; let $\Omega$, referred to $Oxy$, be $\bar{z}(\equiv \bar{x}+i\bar{y})$, and let $\phi$ be the inclination of $\Omega\xi$ to $Ox$. We regard $\phi$ as a variable parameter, and $\bar{z}(\phi)$ as a single-valued function of $\phi$.

Dually, if $O$, referred to the axes $\Omega\xi\eta$, be $\bar{\zeta}(\phi)$, we have

$$\bar{z}+\bar{\zeta}\,e^{i\phi}=0.$$

We introduce the operators

$$\partial \equiv d/d\phi+i, \quad \text{and} \quad \partial' \equiv d/d\phi-i,$$

reducing respectively, when applied to a quantity independent of $\phi$, to

$$\partial_0 \equiv i \quad \text{and} \quad \partial'_0 \equiv -i.$$

We may apply either of these operators $n$ times, writing then, for example, $\partial^n$, and reducing to $\partial_0^n$ when applied to a quantity independent of $\phi$.

Further, we may write the operations of differentiation, with respect to $\phi$, as

$$d\bar{z}/d\phi = \bar{z}^{(1)}, \quad d^2\bar{z}/d\phi^2 = \bar{z}^{(2)}, \quad \ldots, \quad d^n\bar{z}/d\phi^n = \bar{z}^{(n)}, \quad \ldots.$$

2·2. *The cardinal points of the first set.* Let $P$ be a point $z(\equiv x+iy)$, $\zeta(\equiv \xi+i\eta)$, referred to the two sets of axes respectively, and, in the first place, fixed relative to neither set. Then

$$z = \bar{z}+\zeta\, e^{i\phi}, \tag{2·1}$$

and

$$\zeta = \bar{\zeta}+z\, e^{-i\phi}. \tag{2·2}$$

Differentiating (2·1) $n$ times with respect to $\phi$, we obtain

$$z^{(n)} = \bar{z}^{(n)}+e^{i\phi}\,\partial^n\zeta.$$

If $P$, denoted now by $P_\varpi$, be fixed in the plane $\varpi$, we have

$$z^{(n)} = \bar{z}^{(n)}+e^{i\phi}\,\partial_0^n\zeta; \tag{2·3}$$

for a given value of $\phi$ there is one, and only one, point $\mathscr{A}_n(z_n, \zeta_n)$ coinciding with a point $P_\varpi$ for which $z^{(n)} = 0$, and $\mathscr{A}_n$ is given by

$$0 = \bar{z}^{(n)}+e^{i\phi}\,\partial_0^n\zeta_n. \tag{2·4}$$

Then, from (2·1),
$$z_n = \bar{z}-\partial_0'^n\,\bar{z}^{(n)},$$
and, from (2·2),
$$\zeta_n = \partial_0'^n\,\partial^n\bar{\zeta};$$
so that $z_n$ and $\zeta_n$ are, in general, functions of $\phi$.

Further, from (2·3) and (2·4), for the general point $P_\varpi$,

$$z^{(n)} = e^{i\phi}\,\partial_0^n(\zeta-\zeta_n) = \partial_0^n(z-z_n),$$

or

$$z^{(n)} = \partial_0^n\mathscr{A}_nP_\varpi, \tag{2·5}$$

a vector relation. And it will be observed that the point $\mathscr{A}_n$ is fixed neither in $\varpi$, nor in $p$; but, for the value $\phi$, is coincident with that one point $P_\varpi$ for which, relative to $p$, $z^{(n)} = 0$.

Also, from (2·5), for the general point $P_\varpi$ corresponding small variations $\Delta z$ and $\Delta\phi$ are related as follows,

$$\Delta z = \sigma_1\Delta\phi+\sigma_2(\Delta\phi)^2+\ldots+\sigma_n(\Delta\phi)^n+\ldots, \tag{2·6}$$

where the $\sigma$-coefficients are complex, and are given by

$$n!\,\sigma_n = \partial_0^n\mathscr{A}_nP_\varpi.$$

Corresponding to variation of the parameter $\phi$, each point $P_\varpi$ has a path relative to the plane $p$, given by $z(\phi)$, and all the properties of this path, for given $\phi$—the direction of the tangent, the centre of curvature, and the centres of curvature of the successive evolutes—are completely determined by (2·6); that is to say, these properties are completely determined by the position of $P_\varpi$ relative to the enumerable set of points $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \ldots$. Or, the configuration of the points $\mathscr{A}_n$ determines the 'generalized path', at $\phi$, of the plane $\varpi$ relative to the plane $p$—that is, the particular relative displacement of $\varpi$ and $p$; and, conversely, the configuration of the points $\mathscr{A}_n$ is itself completely determined by this relative displacement.

We refer to the points $\mathscr{A}_1, \mathscr{A}_2, \ldots \mathscr{A}_n, \ldots$ as the *cardinal points of the first set*, of the generalized displacement of $\varpi$ relative to $p$.

2·3. In passing, and as a simple example of the use of the cardinal points, we notice that if the ratio of the first two coefficients, in the power series (2·6) in $\Delta\phi$, of the preceding paragraph, be purely real, the point $P_\varpi$ is at an inflexion of its path relative to the plane $p$; or, $P_\varpi$ has three consecutive positions in line, relative to $p$, and all lines so arising pass through the point $\mathscr{A}_2$. And then also, since $\sigma_2/\sigma_1$ is purely real, $P_\varpi$ lies upon the circle having $\mathscr{A}_1\mathscr{A}_2$ as diameter; hence the name, *circle of inflexions*.

If, further, the ratio of the third coefficient to the first, and so also to the second, be purely real, the path of $P_\varpi$ has four consecutive positions in line, relative to the plane $p$. This is then the *Ball point B*, first indicated by Sir R. Ball, and, from (2·6), it appears here as the foot of the perpendicular from $\mathscr{A}_2$ upon the line $\mathscr{A}_1\mathscr{A}_3$. In general, for unrestricted $\mathscr{A}_n$, this is the highest-order contact possible with any straight line fixed in $p$; but if, in addition, the ratio $\sigma_3/\sigma_1$ be purely real—implying that $\mathscr{A}_2\mathscr{A}_4$ is perpendicular to $\mathscr{A}_1\mathscr{A}_3$—the Ball point indicates five consecutive positions in line, relative to $p$. And each additional higher-order contact implies that an additional cardinal point $\mathscr{A}_n$ lies either upon $\mathscr{A}_1\mathscr{A}_3$ or upon $\mathscr{A}_2\mathscr{A}_4$, and conversely.

## 3. The cardinal points of the second set

We have been considering the displacement of the plane $\varpi$ relative to the plane $p$; but, clearly, there is a duality, and we may similarly consider the displacement of $p$ relative to $\varpi$. We investigate, then, the displacement, relative to $\varpi$, of a point $P_p$, fixed in $p$.

From (2·2), we have
$$\zeta = \bar{\zeta} + z\,\mathrm{e}^{-i\phi} \tag{3·1}$$
for a general point $P$, fixed in neither plane; whence, differentiating $n$ times with respect to $\phi$,
$$\zeta^{(n)} = \bar{\zeta}^{(n)} + \mathrm{e}^{-i\phi}\,\partial'^n z. \tag{3·2}$$
If now $P$, denoted by $P_p$, be fixed in $p$,
$$\zeta^{(n)} = \bar{\zeta}^{(n)} + \mathrm{e}^{-i\phi}\,\partial_0'^n z, \tag{3·3}$$
and, for a given value of $\phi$, there is one, and only one, point $\mathscr{A}'_n(z'_n, \zeta'_n)$, coinciding with a point $P_p$ for which $\zeta^{(n)} = 0$, and $\mathscr{A}'_n$ is given by
$$0 = \bar{\zeta}^{(n)} + \mathrm{e}^{-i\phi}\,\partial_0'^n z'_n. \tag{3·4}$$
Whence
$$z'_n = \partial_0^n \partial'^n \bar{z} \quad \text{and} \quad \zeta'_n = \bar{\zeta} - \partial_0^n \bar{\zeta}^{(n)};$$
and these results may be compared with those of §2, as following from the dual character of the relative motion, or displacement. Further, from (3·3) and (3·4), for any point $P_p$,
$$\zeta^{(n)} = \mathrm{e}^{-i\phi}\,\partial_0'^n(z - z'_n) = \partial_0'^n(\zeta - \zeta'_n),$$
or
$$\zeta^{(n)} = \partial_0'^n \mathscr{A}'_n P_p.$$
Connecting corresponding variations $\Delta\zeta$ and $\Delta\phi$ there is a relation similar to (2·6), namely
$$\Delta\zeta = \sigma'_1\,\Delta\phi + \sigma'_2(\Delta\phi)^2 + \ldots + \sigma'_n(\Delta\phi)^n + \ldots,$$
where the $\sigma'$ coefficients are complex, and are given by
$$n!\,\sigma'_n = \partial_0'^n \mathscr{A}'_n P_n.$$
And it follows that all the properties, for given $\phi$, of the path of $P_p$, relative to $\varpi$, depend only upon the relation between $P_p$ and the enumerable set of points $\mathscr{A}'_1, \mathscr{A}'_2, \ldots, \mathscr{A}'_n, \ldots$, and conversely.

In particular, there is a dual inflexion circle, upon $\mathscr{A}'_1\mathscr{A}'_2$ as diameter, and upon this a dual Ball point $B'$, the foot of the perpendicular from $\mathscr{A}'_2$ upon $\mathscr{A}'_1\mathscr{A}'_3$.

We have defined, then, a second enumerable set of *cardinal points* $\mathscr{A}'_1, \mathscr{A}'_2, \ldots, \mathscr{A}'_n, \ldots$, dual to the first set; the configuration of which determines, and is itself determined by, the generalized displacement of $p$ relative to $\varpi$.

Moreover, the points $\mathscr{A}'_n$ are, in general, fixed neither in $p$ nor in $\varpi$.

We have now two enumerable sets of *cardinal points*, $\mathscr{A}_n$ and $\mathscr{A}'_n$; but, clearly, either set determines the other set. Indeed, writing $n = 1$, it is evident that $\mathscr{A}_1$ and $\mathscr{A}'_1$ coincide—at the point commonly known as the instantaneous centre; but, for $n > 1$, $\mathscr{A}_n$ and $\mathscr{A}'_n$ do not coincide, in general.

In this way we have analyzed the generalized displacement of either plane, relative to the other plane, into one or other of two enumerable sets of cardinal points $\mathscr{A}_n$ and $\mathscr{A}'_n$; these, for reference, are given by the following:

$$\mathscr{A}_n(z_n, \zeta_n): \quad z_n = \bar{z} - \partial_0'^n \bar{z}^{(n)}, \quad \zeta_n = \partial_0'^n \partial^n \bar{\zeta};$$

$$\mathscr{A}'_n(z'_n, \zeta'_n): \quad z'_n = \partial_0^n \partial'^n \bar{z}, \quad \zeta'_n = \bar{\zeta} - \partial_0^n \bar{\zeta}^{(n)}.$$

## 4. The paths of the cardinal points and certain associated curves

We are concerned here with the configuration of the cardinal points, and with the derived points and curves, for a single value of the parameter $\phi$; but these points, and these curves, depend upon $\phi$, and for variation of the parameter they have loci, and envelopes, in each of the planes $\varpi$ and $p$. For example, each cardinal point, of each set, has two loci, one in each plane. As a simple illustration, by differentiation with respect to $\phi$, we have the equation

$$e^{i\phi}\,\zeta_n^{(1)} = z_n^{(1)} - z_{n-1}^{(1)},$$

giving a relation between the two loci of $\mathscr{A}_n$. In particular, if $n = 1$,

$$e^{i\phi}\,\zeta_1^{(1)} = z_1^{(1)};$$

and $\mathscr{A}_1, \mathscr{A}'_1$ coincide. That is, we have the familiar rolling property of the centrodes. And for $n > 1$ we have other properties of the 'centrodes' of higher orders.

Further, we have $$z_n^{(1)} = \partial_0(z_{n+1} - z_1),$$

and, for $n = 1$, $$z_1^{(1)} = \partial_0(z_2 - z_1),$$

so that the inflexion circle, and similarly also the dual inflexion circle, touches the centrodes at $\mathscr{A}_1(\mathscr{A}'_1)$. Moreover, these two circles are of equal radii, since

$$z_2 - z_1 = z'_1 - z'_2 \quad \text{and} \quad z_1 = z'_1,$$

and they touch externally at $\mathscr{A}_1(\mathscr{A}'_1)$.

Also, if we write $$\sigma\mathscr{A}_1\mathscr{A}_3 = \mathscr{A}_1\mathscr{A}_2 \quad \text{or} \quad \sigma = (z_1 - z_2)/(z_1 - z_3),$$

and $\tilde{\sigma}$ denote the conjugate of $\sigma$, we have, for the Ball point $Z(\phi)$

$$Z(\phi) = z_1 + \tfrac{1}{2}(\sigma + \tilde{\sigma})(z_3 - z_1),$$

and a corresponding expression for the dual Ball point. Each of these points, then, has two loci, as $\phi$ varies, giving four *Ball lines*, dually related in pairs.

Each Ball line appears also as part of the envelope of the corresponding inflexion circle in the plane considered, the remaining part of the envelope being the corresponding centrode.

Similarly, the *Burmester points* and their duals, subsequently introduced, have loci in the two planes—the several *Burmester lines*, as we may name them; and these loci are parts of the envelopes of the Burmester cubic curves, introduced later.

## 5. The relation between the sets of cardinal points

Each set of cardinal points $\mathscr{A}_n, \mathscr{A}'_n$ determines the other set, and, from §3, the relations between the sets may be expressed in the dual forms

$$z'_n = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} z_r \quad \text{and} \quad \zeta_n = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \zeta'_r.$$

Hitherto, the origins and the axes of co-ordinates have been chosen arbitrarily in each plane; but, since we are interested here in the curves and configurations associated with a single value of the parameter $\phi$, we may take $O$, $\Omega$ and $\mathscr{A}_1$ to be coincident, without loss of generality and also write $\phi = 0$, $\mathscr{A}_1\mathscr{A}_2$ being the coincident axes of $y$ and $\eta$. Then the relations between the co-ordinates of the first few cardinal points, $\mathscr{A}_n(x_n, y_n)$ and $\mathscr{A}'_n(x'_n, y'_n)$, are given by the following, which we may note for subsequent use:

$$\begin{aligned} x'_1 &= 0 = x_1, & y'_1 &= 0 = y_1, \\ x'_2 &= 0 = x_2, & y'_2 &= -y_2, \\ x'_3 &= x_3, & y'_3 &= y_3 - 3y_2, \\ x'_4 &= -x_4 + 4x_3, & y'_4 &= -y_4 + 4y_3 - 6y_2. \end{aligned}$$

## 6. Certain particular configurations of the cardinal points

In passing, it is of interest to examine briefly the configurations of the cardinal points in certain particular relative motions of the planes $\varpi$ and $p$.

Thus, if the displacement of $\varpi$, relative to $p$, be one of pure rotation, that is, if there be a point $P$ fixed both in $\varpi$ and also in $p$, then $P$ may be taken as the base point; and it is evident that all the cardinal points, of each set, coincide with $P$.

If a point $P_\varpi$, of $\varpi$, describe a straight line $l$ in $p$, then from the formulae of the preceding paragraphs, it follows that $\mathscr{A}_2, \mathscr{A}_4, \ldots, \mathscr{A}_{2n}, \ldots$ lie upon $l$, and that $\mathscr{A}_1, \mathscr{A}_3, \ldots, \mathscr{A}_{2n+1}, \ldots$ lie upon the line through $P_\varpi$ perpendicular to $l$. A particular case of this arises if a circle, fixed in $\varpi$, roll, without sliding, along a straight line fixed in $p$; then $\mathscr{A}_1$ is at the point of contact, while $\mathscr{A}_2, \mathscr{A}_3, \ldots, \mathscr{A}_n, \ldots$ coincide with the centre of the circle. The cardinal points of the second set lie upon the radius (produced) to the point of contact, being equally spaced along this line, at distances equal to the radius of the circle. If the radius of this circle tend to zero, and the motion become one of pure rotation, all the cardinal points, of each set, tend to coincidence at $\mathscr{A}_1$, the point of contact.

Another case of interest arises when, in addition, a second point $P'_\varpi$, of $\varpi$, describes a second straight line $l'$, in $p$; then $\mathscr{A}_1(\mathscr{A}'_1)$ is given by the intersection of the respective perpendiculars to $l$ and $l'$, at $P_\varpi$ and $P'_\varpi$, and $\mathscr{A}_1, \mathscr{A}_3, \ldots, \mathscr{A}_{2n+1}, \ldots$ coincide, while $\mathscr{A}_2, \mathscr{A}_4, \ldots \mathscr{A}_{2n}, \ldots$ coincide with the point of intersection of the lines $l$ and $l'$. The cardinal points of the second set $\mathscr{A}'_n$ all lie upon $\mathscr{A}_2\mathscr{A}_1$ produced, so that $\mathscr{A}_2\mathscr{A}'_n = 2^{n-1}\mathscr{A}_2\mathscr{A}'_1$. The displacement of $\varpi$ relative to

$p$, in this case, is the so-called *elliptical* displacement, while that of $p$ relative to $\varpi$, the dual, is *cardioid* displacement.

Other types, for example, *conchoidal* displacement, may be examined; but these cases are, of course, highly specialized, and they give rise to very degenerate forms of many of the curves and configurations of the succeeding paragraphs.

More generally, let a point $P_\varpi$, of $\varpi$, describe a given curve $C$, fixed in $p$; then the cardinal points are only partially determined, since the relative motion itself is not completely determined—owing to the unspecified correlation between the parameter $\phi$ and the curve $C$. But, if a second point $P'_\varpi$, of $\varpi$, be constrained to describe a second curve $C'$, fixed in $p$, the cardinal points are completely determined, and their several positions may be found from knowledge of the two curves $C$ and $C'$; this involves the analysis and investigation of the following paragraphs.

In particular, if $C$ and $C'$ be circles, we are dealing with the important special case of three-bar displacement, of the plane $\varpi$ relative to the plane $p$.

## 7. The centres of curvature of the path and of its evolutes

In the general case, let $P_\varpi$ be a point of $\varpi$, and let the tangent at $P_\varpi$, to its path relative to $p$, be inclined at angle $\psi$ with a line fixed in $p$. Then, for this point, $z$ may be regarded as a function either of $\psi$ or of $\phi$; and the centre of curvature $C(Z)$ of the path is given by

$$Z = z + i\,dz/d\psi = \Omega z;$$

where

$$\Omega \equiv 1 + i\,d/d\psi \equiv i\delta',$$

and

$$\delta' \equiv d/d\psi - i.$$

Repeated application of the operator $\Omega$ gives the centre of curvature, $C_n(Z_n)$, of the $n$th evolute of the path of $P_\varpi$

$$Z_n = \Omega^{n+1} z,$$

and the vector radius of curvature, $\rho_n$, of the $n$th evolute, is given by

$$\rho_n = \Omega^n(\Omega - 1)\,z;$$

so that, $\rho$ being the radius of curvature of the path of $P_\varpi$, we have

$$\rho_1 = \Omega\rho, \quad \rho_2 = \Omega^2\rho, \quad \ldots, \quad \rho_n = \Omega^n\rho, \quad \ldots.$$

Further, the vector displacement of the centre of curvature of the $(n-1)$th evolute, from the $n$th cardinal point $\mathscr{A}'_n$, of the second set, is given by

$$\mathscr{A}'_n C_{n-1} = \partial_0^n(\delta'^n - \partial^n)\,z,$$

taking $z(\equiv \bar{z})$ as base point.

If $\rho_1 = 0$, then $P_\varpi$ is at a point of stationary curvature of its path relative to $p$, and such points, for given $\phi$, lie upon the curve

$$\Omega(\Omega - 1)\,z = 0,$$

a cubic curve of some importance, which we consider subsequently.

These results give the centre of curvature of the path of the general point $P_\varpi$, and of the successive evolutes of this path, in terms of the configuration of the cardinal points. For they involve $\psi$ as a function of $\phi$, say $\psi = f(\phi)$, and the function $f(\phi)$, being a property of the path of $P_\varpi$, for given $\phi$, may be expressed readily in terms of the position of $P_\varpi$ relative to the cardinal points $\mathscr{A}_n$ and $\mathscr{A}'_n$, $n = 1, 2, 3, \ldots$.

## 8. Orthogonal sets of cardinal points

In passing, we may notice a particular case of interest. If there be a point $P_\varpi$ for the relative path of which

$$\psi = \phi + \text{const.},$$

this relation being differentiable repeatedly, we may take $P_\varpi$ as base point, and then the operators $\delta'$ and $\partial'$ are identical. The cardinal points $\mathscr{A}'_n$ of the second set coincide successively with the centres of curvature of the several evolutes of the path of $P_\varpi$. We remark that this special case arises when an arbitrary curve, fixed in $p$, is throughout the displacement touched by a straight line fixed in $\varpi$, the point of contact being fixed in the line.

Then $\mathscr{A}'_1(\mathscr{A}_1)$ is the centre of curvature of the relative path of $P_\varpi$, $\mathscr{A}'_2$ that of the first evolute, and in general $\mathscr{A}'_{n+1}$ that of the $n$th evolute. Here $\mathscr{A}'_n\mathscr{A}'_{n+1}$ is perpendicular to $\mathscr{A}'_{n+1}\mathscr{A}'_{n+2}$ for all values of the positive integer $n$; and such a set of cardinal points, $\mathscr{A}'_n$, may be described as an *orthogonal* set.

It is the case that if a set of cardinal points be orthogonal, for a single value of the parameter $\phi$, the set is orthogonal for all values of $\phi$. For, from the relations of § 3, defining these points, we have

$$\zeta_n'^{(p)} - \zeta_{n-1}'^{(p)} = \partial_0'^p(\zeta'_{n+p} - \zeta'_{n+p-1}),$$

$p$ being a positive integer. If now, for the value $\phi$, the set be orthogonal,

$$\zeta_n'^{(p)} - \zeta_{n-1}'^{(p)} = \nu_p(\zeta'_n - \zeta'_{n-1}),$$

where $\nu_p$ is real. The line $\mathscr{A}'_n\mathscr{A}'_{n-1}$ remains fixed in direction, therefore, throughout the displacement, and the set $\mathscr{A}'_n$ remains orthogonal.

Orthogonality of one of the sets of cardinal points is characteristic of the type of motion described above, and conversely.

Clearly orthogonality of one set of cardinal points is incompatible with orthogonality of the dual set.

## 9. The general configuration of the cardinal points

9·1. In general, the relation of § 8 is not satisfied for any point $P_\varpi$; we proceed now, for quite general displacement of $\varpi$ relative to $p$, to find the centres, and the radii, of curvature of the relative path of $P_\varpi$, and of the successive evolutes, in terms of the configuration of the cardinal points.

Henceforward, we omit the suffix $\varpi$ in $P_\varpi$, since, unless otherwise indicated, we deal only with a point $P$, fixed in $\varpi$.

From § 7, the radius of curvature of the path of $P$—the vector $PC$—is given by

$$\rho = (i/\theta)\, z^{(1)},$$

where $$\theta = d\psi/d\phi;$$

or, from (2·5), $$\theta\rho = P\mathscr{A}_1.$$

For the vector radius of curvature, $\rho_n$, of the $n$th evolute of this path, we write

$$\theta^{2n+1}\rho_n = i^n\omega(n)\, P\mathscr{A}_1,$$

where $\omega(n)$ is real. We write also $$P\mathscr{A}_n = (\xi_n + i\eta_n)\, P\mathscr{A}_1,$$

and, since from (2·5) $$\frac{d}{d\phi}(P\mathscr{A}_n) = iP\mathscr{A}_{n+1},$$

we have
$$\xi'_n = -\eta_{n+1}+\xi_2\eta_n+\xi_n\eta_2,$$
$$\eta'_n = \xi_{n+1}-\xi_2\xi_n+\eta_2\eta_n,$$
where the prime (′) now denotes, temporarily, differentiation with respect to the parameter $\phi$. Further, with the axes and notation of § 5, we have
$$\xi_n = 1-(xx_n+yy_n)/r^2 \quad \text{and} \quad \eta_n = (x_n y - y_n x)/r^2,$$
where
$$\mathscr{A}_1 P = r = +\sqrt{(x^2+y^2)}.$$

Then, since
$$\rho_{n+1} = \Omega\rho_n,$$
we have
$$\theta = \xi_2,$$
and also the mixed equation
$$\omega(n+1) = \xi_2\omega'(n) - \{\xi_2\eta_2+(2n+1)\,\xi'_2\}\,\omega(n). \tag{9·1}$$
This latter equation may be written
$$\omega(n+1) = \xi_2\omega'(n)+\Omega(n)\,\omega(n) = E_n\omega(n),$$
where
$$\Omega(n)+\xi_2\eta_2+(2n+1)\,\xi'_2 = 0, \quad \Omega(n) = \Omega(0)-2n\xi'_2,$$
and the operator $E_n$ is defined by
$$E_n \equiv \xi_2\, d/d\phi+\Omega(n).$$
Thus, symbolically, we may write the solution of (9·1)
$$\omega(n) = \left(\prod_{\nu=n-1}^{0} E_\nu\right)\omega(0) = \prod_{n-1}^{0} E_\nu \,.\, 1,$$
since $\omega(0) = 1$. We have also
$$\xi'_2 = 2\xi_2\eta_2-\eta_3.$$
The solution of the mixed equation gives immediately the radius of curvature $\rho_n$ of the $n$th evolute of the relative path of the general point $P$, in terms of the relationship between $P$ and the cardinal points $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n+2}$, of the first set.

9·2. It is evident that, if $\xi_2 = 0 = \eta_3$, that is, if $P$ coincide with the Ball point $B$, then $\omega(n) = 0$ for all values of $n$ greater than unity; also, if $\xi_\mu = 1$, $\eta_\mu = 0, \mu > 1$, then again $\omega(n) = 0$, $n > 1$. The latter case arises if the cardinal points $\mathscr{A}_n$ coincide, that is, if we are dealing with a pure rotation.

Clearly, the function $\omega(n)$, and also the function $\omega'(n)$, are polynomials in the various $\xi_\mu, \eta_\mu$; and, if we regard $\xi_{\mu+1}$ and $\eta_{\mu+1}$ as of 'weight' $\mu$, then $\omega(n)$ is throughout of weight $2n$ and, in this sense, is homogeneous. Also, each function is linear in the quantity $\xi_\mu, \eta_\mu$ of greatest weight.

Further, we have
$$\omega(n) = \sum_{p=1}^{2n} (u_p/r^{2p}),$$
where $u_p$ is homogeneous, and of degree $p$, both in $x, y$, and also in the $x_\mu, y_\mu$; $\omega(n)$ is, of course, a pure number.

We write
$$w(n) = r^{4n}\omega(n),$$
and consider the curves
$$w(n) = 0 \quad \text{or} \quad \sum_{p=1}^{2n} u_p r^{2(2n-p)} = 0 \quad (n = 1, 2, \ldots).$$

Each curve, for the appropriate value of $n$, gives the aggregate of points for which

$$\rho_n = 0.$$

The curve is circular, of degree $4n-1$, having multiple points of order $2n-1$ at the circular points; and it passes through the first cardinal point $\mathscr{A}_1$, having there a multiple point of order $2n$. Also, for all values of $n > 0$, it passes through the point given by $\xi_2 = 0$, $\eta_3 = 0$, that is, the Ball point, the foot of the perpendicular from $\mathscr{A}_2$ upon $\mathscr{A}_1\mathscr{A}_3$.

The preceding curves correspond to a single value of the parameter $\phi$; but we notice in passing, that if

$$\omega(n+1) = 0 \quad \text{and} \quad \omega(n) = 0,$$

then, in general

$$\omega'(n) = 0.$$

The envelope then of the curve $w(n) = 0$, for varying $\phi$, is given by the loci of the points of intersection of the curves

$$w(n+1) = 0 \quad \text{and} \quad w(n) = 0.$$

9·3. The curves $w(n) = 0$ have fixed intersections at the circular points, at the first cardinal point $\mathscr{A}_1$, and at the Ball point; and, in addition, they have certain 'free' intersections, depending upon the configuration of the cardinal points. In particular, interest centres in the free intersections of the curves

$$w(n) = 0 \quad \text{and} \quad w(1) = 0;$$

more particularly still, in the possibility of coincidences between the sets of intersections with $w(1) = 0$, for the several values of $n$. For example, such coincidences arise, for all values of $n$, in the important special case of three-bar displacement.

A single free intersection, common to all the curves $w(n) = 0$, implies some limitation upon the set of cardinal points, and the addition of a second free intersection, common to all these curves, implies that the configuration of the cardinal points is completely determined; we are dealing then with three-bar displacement of the plane $\varpi$, relative to the plane $p$.

For unrestricted $\mathscr{A}_n$ we have

$$\begin{aligned}
\omega(0) &= 1,\\
\omega(1) &= \eta_3 - 3\xi_2\eta_2,\\
\omega(2) &= 3\omega^2(1) + 6\xi_2\eta_2\omega(1) + \xi_2(\xi_4 - 4\xi_2\xi_3 + 3\xi_2^3 + 3\xi_2\eta_2^2),\\
\omega(3) &= 15\omega^3(1) + 60\xi_2\eta_2\omega^2(1) + \xi_2(10\xi_4 - 30\xi_2\xi_3 + 11\xi_2^3 + 75\xi_2\eta_2^2)\,\omega(1)\\
&\quad + \xi_2^2(-\eta_5 + 5\xi_2\eta_4 + 10\eta_2\xi_4 - 30\xi_2\eta_2\xi_3 + 30\xi_2\eta_2^3),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

In the sequel, we shall be interested more especially in the aggregate of points $P$ for which

$$\rho = 0, \quad \text{that is,} \quad \omega(1) = 0.$$

Such points lie upon a certain cubic curve, and, for these points, the mixed equation (9·1) takes the form

$$\xi_2^{-1}\omega(n+1) = \omega'(n) + 2n\eta_2\omega(n),$$

and then, in particular, we have

$$\begin{aligned}
\xi_2^{-1}\omega(2) &= \xi_4 - 4\xi_2\xi_3 + 3\xi_2(\xi_2^2 + \eta_2^2),\\
\xi_2^{-2}\omega(3) &= -\eta_5 + 5\xi_2\eta_4 + 10\xi_4\eta_2 - 30\xi_2\eta_2(\xi_3 - \eta_2^2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

An important special case arises if $\rho \neq 0$, $\rho_1 = 0$, and, in addition, $\rho_2 = 0 = \rho_3 = \ldots = \rho_n$; for then the relative path of $P$ has contact of order $n+3$ with its circle of curvature. This was investigated, at some length, by Tschebycheff for small values of $n$. Here we have, from § 9·2,

$$\omega'(1) = 0 = \omega'(2) = \ldots = \omega'(n+1),$$

and these relations imply some limitation upon the configuration of the cardinal points. For then

$$\xi_2^{-1}\eta_3 = 3\eta_2,$$
$$\xi_2^{-1}\xi_4 = 4\xi_3 - 3(\xi_2^2 + \eta_2^2),$$
$$\xi_2^{-1}\eta_5 = 5\eta_4 + 10\eta_2(\xi_3 - 3\xi_2^2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

That is, the cardinal points $\mathscr{A}_3, \mathscr{A}_4, \ldots, \mathscr{A}_{n+2}$, of the first set, are restricted alternately to lines parallel to, and perpendicular to, the line $P\mathscr{A}_1$.

## 10. The *Rückkehrpole* and the *Wendepole*

Hitherto, we have regarded the plane $\varpi$ as an aggregate of points $P_\varpi$, and similarly for the plane $p$. But, instead, we may regard $\varpi$ as an aggregate of lines $l_\varpi$, fixed in $\varpi$; and, in the displacement corresponding to a change in the parameter $\phi$, each such line has an envelope in the plane $p$.

Let $q$ and $q_0$ be the perpendiculars, from $O$ and $\Omega$ respectively, upon a line $l_\varpi$, so that $q(\phi)$ is a function of $\phi$, and let $\alpha$ and $\alpha_0$ be their inclinations to $Ox$ and $\Omega\xi$; then

$$\alpha = \phi + \alpha_0,$$

and $\alpha_0$ is independent of $\phi$. The equation of $l_\varpi$, referred to the axes $Oxy$, may be written

$$l_\varpi \equiv x\cos\alpha + y\sin\alpha - q = 0.$$

The point of contact, $\mathscr{Z}(\phi)$, of this line, with its envelope, for varying values of $\phi$, is given by

$$\mathscr{Z}(\phi) = e^{i\alpha}\,\partial_0\,\partial' q,$$

which, in general, is fixed in neither plane. Hence, the centre of curvature, $Z(\phi)$, of this envelope is given by

$$Z(\phi) = e^{i\alpha}\partial_0^2\partial' Dq,$$

and the centre of curvature, $Z_n(\phi)$, of the $n$th evolute of the envelope, by

$$Z_n(\phi) = e^{i\alpha}\partial_0^{n+2}\partial' D^{n+1}q,$$

where $D = d/d\phi$.

If now $\tilde{\bar{z}}(\phi)$ be the conjugate of $\bar{z}(\phi)$, we have

$$2q = e^{-i\alpha}\bar{z} + e^{i\alpha}\tilde{\bar{z}} + 2q_0,$$

and if, further, $\tilde{\mathscr{A}}_n'(\tilde{z}_n')$ be the conjugate of $\mathscr{A}_n'(z_n')$, so that

$$\tilde{z}_n' = \partial_0'^n\partial^n\tilde{\bar{z}},$$

while, as previously,

$$z_n' = \partial_0^n\partial'^n\bar{z},$$

it follows, from the preceding, that

$$2Z(\phi) = z_1' + z_2' - e^{2i\alpha}(\tilde{z}_1' - \tilde{z}_2'), \qquad (10\cdot1)$$

and, more generally, that

$$2Z_n(\phi) = z'_{n+1} + z'_{n+2} \pm e^{2i\alpha}(\tilde{z}'_{n+1} - \tilde{z}'_{n+2}), \tag{10·2}$$

according as $n$ is odd, or even.

It is evident that the envelopes of lines of $\varpi$, parallel to $l_\varpi$, are themselves parallel curves, and, at corresponding points, have coincident centres of curvature, and also coincident centres of curvature for the several evolutes. Further, varying $\alpha$ for given $\phi$ we may consider the centres of curvature of the envelopes, and of their evolutes, of all lines of $\varpi$; and, from (10·1), all centres of curvature, of such envelopes, for given $\phi$ and varying $\alpha$, lie upon the circle having $\mathscr{A}'_1\mathscr{A}'_2$ as diameter. More generally, from (10·2), the centres of curvature of all the $n$th evolutes lie upon the circle having $\mathscr{A}'_{n+1}\,\mathscr{A}'_{n+2}$ as diameter. And it follows that the normals to the envelopes, and to the successive evolutes, pass respectively through the cardinal points $\mathscr{A}'_1, \mathscr{A}'_2, \ldots \mathscr{A}'_n, \ldots$ of the second set.

There are clearly dual properties, involving similarly the cardinal points $\mathscr{A}_n$ of the first set, and the envelopes, in $\varpi$, of lines fixed in $p$.

*It is evident that the cardinal points $\mathscr{A}'_n$ and $\mathscr{A}_n$ coincide respectively with the *Rückehrpole* and the *Wendepole* of earlier German writers; but these cardinal points play a more fundamental role in plane kinematics than is indicated by this property.

## 11. The circles of inflexion and the Ball points

The tangent to the relative path of a quite general point $P$ is determined by the first cardinal point $\mathscr{A}_1$, being perpendicular to $\mathscr{A}_1 P$; the radius of curvature of the path by the addition of the second cardinal point $\mathscr{A}_2$, that of the first evolute by the further addition of the third cardinal point $\mathscr{A}_3$, and so on indefinitely. And we may verify, quite readily, that if there be a point $P$, for the path of which $\phi = \psi + \text{const.}$, this relation being differentiable repeatedly, the cardinal points $\mathscr{A}'_n$, of the second set, form an orthogonal system.

Further, it is evident that, if $\xi_2 = 0$, that is, if $P$ lies upon the circle having $\mathscr{A}_1\mathscr{A}_2$ as diameter, the corresponding path curvature is, in general, zero; hence the name *circle of inflexions*. Or, the point $P$ has (at least) three consecutive positions in line, relative to the plane $p$, and all lines so arising pass through $\mathscr{A}_2$. The point $\mathscr{A}_1$, upon the circle $\xi_2 = 0$, is a singular point of the function $\rho$, the radius of curvature of the path.

If, in addition, $P$ lies upon the line $\mathscr{A}_1\mathscr{A}_3$, then $\eta_3 = 0$, so that $P$ lies also upon the curve $w(1) = 0$, the aggregate of points for which $\rho_1 = 0$; and $P$ coincides with the Ball point $B$, having four consecutive positions in line. The Ball point arises then as the second intersection of the line $\mathscr{A}_1\mathscr{A}_3$ with the inflexion circle; or as the foot of the perpendicular from $\mathscr{A}_2$ upon $\mathscr{A}_1\mathscr{A}_3$. And it gives, for the quite general relative motion, the closest approximation to straight-line displacement, of all points $P_\varpi$, fixed in $\varpi$.

If, as a special case, the Ball point has five consecutive positions in line, some limitation upon the position of the fourth cardinal point $\mathscr{A}_4$ is implied; for then, from the preceding

* The points $\mathscr{A}_2$ and $\mathscr{A}_3$ have also been obtained (Léauté) as the limiting position of the acceleration centre, and of the first hypercentre, as the relative angular acceleration of the two planes tends to zero; and, in fact, the remaining points $\mathscr{A}_4, \mathscr{A}_5, \ldots$ can be derived in like fashion. And dually also the points $\mathscr{A}'_n$. But these points, and their duals, are intrinsically properties of the 'generalized path' of $\varpi$ relative to $p$, and not of any particular motion in this path. And, in any case, their role is more fundamental than is indicated by this property.

paragraphs, $\xi_4 = 0$, so that $\mathscr{A}_2\mathscr{A}_4$ is perpendicular to $\mathscr{A}_1\mathscr{A}_3$. And, as in §2·3, we may proceed, in like manner, to show that each successive higher order contact of the path of $B$ with a straight line of $p$ implies that an additional cardinal point, $\mathscr{A}_n$, of the first set, lies alternately upon one or other of the lines $\mathscr{A}_1\mathscr{A}_3$, $\mathscr{A}_2\mathscr{A}_4$.

There is evidently a dual inflexion circle, upon $\mathscr{A}_1'\mathscr{A}_2'$ as diameter, and upon this a dual Ball point $B'$, the foot of the perpendicular from $\mathscr{A}_2'$ upon $\mathscr{A}_1'\mathscr{A}_3'$. And the two inflexion circles are of equal radii, and touch externally at $\mathscr{A}_1(\mathscr{A}_1')$.

## 12. The Ball point as a singular point of the functions $\rho_n$

The radius of curvature $\rho_n$ $(n \geqslant 1)$, of the $n$th evolute is a function of position, having singularities at the first cardinal point $\mathscr{A}_1$, and at the Ball point $B$. For we have written

$$\xi_2^{2n+1}\rho_n = i^n\omega(n)\, P\mathscr{A}_1,$$

and we have considered the curves $\omega(n) = 0,$

of weight $2n$, for the general point of which, excluding $\mathscr{A}_1$ and $B$, we have

$$\rho_n = 0.$$

The Ball point lies both upon the curve $w(n) = 0$, and also upon the inflexion circle $\xi_2 = 0$; so that for this point $\rho_n$ is undetermined. We may examine the values of $\rho_n$ in the neighbourhood of $B$; this function of position behaves variously as $P$ tends to the point $B$, according to the path followed.

We may consider, for example, the special case $n = 1$, and write $|BP| = R$, so that as $P \rightarrow B$ we have $R \rightarrow 0$.

If $P \rightarrow B$ along a general straight line then $R^2\rho_1 \rightarrow$ a finite limit, and, in particular, if the line coincide with $\mathscr{A}_1\mathscr{A}_3$, the limit is $3\mathscr{A}_2B.\mathscr{A}_1B^2$. If the line be the tangent at $B$ to the curve $w(1) = 0$, then $R\rho_1 \rightarrow$ a finite limit, while if it be the tangent at $B$ to the inflexion circle $R^5\rho_1 \rightarrow$ a finite limit.

On the inflexion circle, in general, $1/\rho_1 = 0$, and on the curve $w(1) = 0$ we have $\rho_1 = 0$, in general; while, if $P \rightarrow B$ along a certain conic touching $w(1) = 0$ at $B$, then $\rho_1 \rightarrow$ a finite limit.

## 13. The Burmester cubic

A point $P$, for the relative path of which the radius of curvature $\rho_1$, of the first evolute, vanishes, has stationary path curvature; that is, the relative path at $P$ has four-point contact with its circle of curvature. From §§9·1, 9·3, such points are given by

$$\theta^{-2}(\theta^{-1}\eta_3 - 3\eta_2)\, P\mathscr{A}_1 = 0.$$

Upon an arbitrary straight line through $\mathscr{A}_1$, the quantities $\eta_2 P\mathscr{A}_1$, $\eta_3 P\mathscr{A}_1$ are constants, being the perpendicular distances, from this line, of the cardinal points $\mathscr{A}_2$ and $\mathscr{A}_3$ respectively; the preceding relation, therefore, determines three points $\theta_1, \theta_2, \theta_3$ upon this line, of which two coincide at $\mathscr{A}_1$. The aggregate of all such points, for all lines through $\mathscr{A}_1$, is then a rational cubic curve, having a node at $\mathscr{A}_1$, the tangents there being $\mathscr{A}_1\mathscr{A}_2$ and a perpendicular line. And these tangents form rectangular natural axes to the system.

We have the curve $w(1) = u_1 r^2 + u_2,$

in the notation of §9·2, where

$$u_1 = y_3' x - x_3' y \quad \text{and} \quad u_2 = 3y_2'^2 xy.$$

If, then, we write $3y_2^2\alpha = y_3 - 3y_2 = y_3', \quad 3y_2^2\beta + x_3 = 0, \quad x_3' = x_3,$
the equation becomes $\Gamma \equiv (\alpha x + \beta y)(x^2+y^2) + xy = 0,$

$\mathscr{A}_1$ being the origin of co-ordinates, and $\mathscr{A}_1\mathscr{A}_2$ the $y$-axis.

The curve $\Gamma$ is a circular cubic, having a node, with perpendicular tangents, at the origin $\mathscr{A}_1$; that is, the curve is a strophoid. And, in this connexion, it first appeared, I believe, in the writings of Burmester.

There are various derivations of the rational cubic curve in a plane but, as related to the present investigation, and as showing the genesis of the curve from the first three cardinal points, we may note the following.

Let any line parallel to $\mathscr{A}_1\mathscr{A}_2$ intersect the inflexion circle in $p$, $q$, and the line $\mathscr{A}_1\mathscr{A}_3$ in $r$; let $\mathscr{A}_1 p$, $\mathscr{A}_1 q$ intersect the line through $r$, parallel to $\mathscr{A}_1\mathscr{A}_3'$, in $P$, $Q$. Then $P$, $Q$ lie upon the Burmester cubic.

Or, let $(D)$, $(E)$ be related ranges upon the lines $\mathscr{A}_1\mathscr{A}_2$, $\mathscr{A}_2\mathscr{A}_3$, respectively, such that $\mathscr{A}_1, \mathscr{A}_2, D_\infty$ correspond to $\mathscr{A}_2, \mathscr{A}_3, \infty$, where $2D_\infty\mathscr{A}_2 = \mathscr{A}_2\mathscr{A}_1$; then the foot of the perpendicular from $D$, upon the line $E\mathscr{A}_1$, lies upon the Burmester cubic.

The curve is determined by the first three cardinal points alone; but there are many $(\infty^1)$ sets of these cardinal points giving the same cubic curve. Indeed, $\mathscr{A}_1$ being given, and $\mathscr{A}_3$ lying upon the parabola

$$(\alpha x + \beta y)^2 + 3\beta x = 0,$$

a suitable point $\mathscr{A}_2$ can always be found, upon the fixed line $\mathscr{A}_1\mathscr{A}_2$, the line $\mathscr{A}_2\mathscr{A}_3$ enveloping the parabola

$$3(\alpha x + \beta y)^2 + 2\beta x = 0;$$

the axes of these parabolas are parallel to the single real asymptote of the cubic $\Gamma$.

It is convenient to change from $\theta$ to $\epsilon$ as the parameter of the general point upon $\Gamma$, where

$$\epsilon - \alpha' + (\alpha' - \alpha)\,\theta = 0,$$

and $\alpha'$ is derived from the cardinal points $\mathscr{A}_2', \mathscr{A}_3'$, of the second set, as $\alpha$ is derived from $\mathscr{A}_2, \mathscr{A}_3$; that is,

$$3y_2'^2\alpha' = y_3' - 3y_2' = y_3, \quad \beta' = \beta.$$

Then the curve is given parametrically by

$$x/-\beta = y/\epsilon = \epsilon/(\epsilon-\alpha)(\epsilon^2+\beta^2).$$

The Ball point, indicating stationary path curvature, is given by $\epsilon = \alpha'$, the single real point at infinity, the circular points, and the node, by $\epsilon = \alpha$, $\epsilon^2+\beta^2 = 0$, $\epsilon = 0, \infty$, respectively. Further, the conditions that three points $\epsilon_1, \epsilon_2, \epsilon_3$ should be collinear, that four points $\epsilon, \epsilon_2, \epsilon_3, \epsilon_4$ should be concyclic, that six points $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$ should lie upon a conic, are respectively

$$\epsilon_1\epsilon_2\epsilon_3 = \alpha\beta^2, \quad \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = \alpha^2\beta^2, \quad \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\epsilon_6 = \alpha^2\beta^4.$$

The curve, being circular, has a double focus $F$; this point is upon the curve, and of parameter given by $\epsilon + \alpha = 0$. It plays some part in the subsequent theory. There are also two other foci $F_1$, $F_2$, the four points of contact of the corresponding tangents from the circular points being given by

$$\epsilon = \pm(\pm i\alpha\beta)^{\frac{1}{2}}.$$

Or, the foci $F$, $F_1$, $F_2$ are given by

$$2iZ(\alpha + i\beta) = 1, \quad iZ\{\surd\alpha \pm \surd(i\beta)\}^2 = 1,$$

respectively.

The cubic intersects the real asymptote in the point $H(\epsilon)$, given by $\alpha\epsilon = \beta^2$, and the line $HF$, the satellite of the line infinity, touches the curve at the double focus $F$.

We notice, in passing, as of use in a cognate investigation, that if the third cardinal point $\mathscr{A}_3$ be allowed to move upon a fixed line $l$, then we have a pencil of cubics, through a fixed point $Q$, where $\mathscr{A}_1 Q$ is parallel to the line $l$.

### 14. The dual cubic

We may consider the centre of curvature $C$ of the relative path of the general point $P$, fixed in the plane $\varpi$; then

$$\xi_2 PC = P\mathscr{A}_1,$$

and we write, temporarily,

$$C\mathscr{A}'_n = (\bar{\xi}_n + i\bar{\eta}_n)\, C\mathscr{A}_1,$$

since $\mathscr{A}_1$ and $\mathscr{A}'_1$ coincide.

Then, from § 5, $\mathscr{A}'_2\mathscr{A}_2 = 2\mathscr{A}_1\mathscr{A}_2$ and $\mathscr{A}'_3\mathscr{A}_3 = 3\mathscr{A}_1\mathscr{A}_2$,

so that we have the vector relations

$$\mathscr{A}'_2C + CP + P\mathscr{A}_2 = 2(\mathscr{A}_1P + P\mathscr{A}_2),$$

and

$$\mathscr{A}'_3C + CP + P\mathscr{A}_3 = 3(\mathscr{A}_1P + P\mathscr{A}_2).$$

The former of these relations gives

$$\left.\begin{aligned} \bar{\xi}_2 + \xi_2 &= 1, \\ \text{and} \quad \bar{\xi}_2\bar{\eta}_2 &= \xi_2\eta_2, \end{aligned}\right\} \tag{14·1}$$

and the latter gives

$$\left.\begin{aligned} \bar{\xi}_2\bar{\xi}_3 + \xi_2\xi_3 + 3\bar{\xi}_2\xi_2 &= 1, \\ \text{and} \quad \bar{\xi}_2\bar{\eta}_3 + \xi_2\eta_3 &= 3\xi_2\eta_2. \end{aligned}\right\} \tag{14·2}$$

Then, if $\bar{\rho}_1$ be the radius of curvature of the first evolute of the path, relative to $\varpi$, of the point $C_p$, fixed in the plane $p$ and instantaneously coincident with $C$, and, as previously, $\rho_1$ be the similar quantity for the relative path of the point $P$, fixed in the plane $\varpi$, we have, from § 9·1,

$$\left.\begin{aligned} \bar{\xi}_2^3\bar{\rho}_1 &= i\omega_1(1)\, C\mathscr{A}_1, \\ \text{and} \quad \xi_2^3\rho_1 &= i\omega(1)\, P\mathscr{A}_1, \end{aligned}\right\} \tag{14·3}$$

where, from § 9·3,

$$\omega_1(1) = \bar{\eta}_3 - 3\bar{\xi}_2\bar{\eta}_2,$$

and

$$\omega(1) = \eta_3 - 3\xi_2\eta_2.$$

Using now the relations (14·1) and (14·2), we have

$$\bar{\xi}_2^4\bar{\rho}_1 P\mathscr{A}_1 = \xi_2^4\rho_1 C\mathscr{A}_1,$$

and, since

$$C\mathscr{A}_1 = CP + P\mathscr{A}_1 = (-1/\xi_2 + 1)\, P\mathscr{A}_1 = -\bar{\xi}_2 P\mathscr{A}_1/\xi_2,$$

we may substitute in (14·3) to obtain $\bar{\xi}_2^3\bar{\rho}_1 = \xi_2^3\rho_1$.

When $P$ lies upon the Burmester cubic, $\rho_1 = 0$, and then, in general, $\bar{\rho}_1 = 0$; so that $C$ lies upon a dual cubic $\Gamma'$, derived from the cardinal points $\mathscr{A}'_1, \mathscr{A}'_2, \mathscr{A}'_3$ of the second set, precisely as the Burmester cubic is derived from the cardinal points $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ of the first set.

We have then, as the curve upon which $C$ lies, a dual circular cubic $\Gamma'$, with a right-angled node at $\mathscr{A}'_1(\mathscr{A}_1)$, the tangents there being the same as those to $\Gamma$. Further, we may regard $P$ and $C$ as *corresponding points*, upon the dual cubics, the relationship being reciprocal; so that $P$ coincides with the centre of curvature of the relative path of $C_p$. The parametric equation of $\Gamma'$ is

$$x/-\beta = y/\epsilon = \epsilon/(\epsilon-\alpha')\,(\epsilon^2+\beta^2),$$

and the Cartesian equation

$$\Gamma' \equiv (\alpha' x+\beta y)\,(x^2+y^2)+xy = 0,$$

the parameters of corresponding points being equal. In particular, we notice that the Ball point upon either curve corresponds to the real point at infinity upon the other curve.

In certain special cases, the Burmester cubic breaks up ; for example, if $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ be collinear, as in several of the particular cases of § 6, then $\beta = 0$, and the cubic breaks up into the line $\mathscr{A}_1\mathscr{A}_2$, together with a circle, touching the inflexion circle at $\mathscr{A}_1$. And similarly for the dual cubic.

## 15. The Burmester points

15·1. For each point upon the Burmester cubic we have $\rho_1 = 0$, so that the corresponding path has four-point contact with its circle of curvature. If, in addition, $\rho_2 = 0$, the path has five-point contact with its circle of curvature; and, in general, apart, that is to say, from some limitation upon the position of the fifth cardinal point $\mathscr{A}_5$, this is the highest order contact possible, for any point $P_\varpi$, with its corresponding circle of curvature. Following Müller we refer to such a point as a *Burmester point*.

Clearly these points arise, in general, from the intersections of the curves $w(2) = 0$ and $w(1) = 0$, of § 9·2. But we may deal with the matter in more general fashion by considering the intersections of the curves $w(n) = 0$ and $w(1) = 0$. These intersections cluster at the multiple point $\mathscr{A}_1$, at the absolute points, and at the Ball point; but there are, in addition, certain 'free' intersections, the positions of which depend upon the configuration of the cardinal points.

The curve $w(n) = 0$ depends upon all the cardinal points up to, and including, $\mathscr{A}_{n+2}$, and, further, depends linearly upon the co-ordinates of $\mathscr{A}_{n+2}$. If then the cardinal points $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n+1}$ be regarded as assigned, the free intersections of the two curves appear as a linear series of sets of points, of freedom two, upon the cubic $w(1) = 0$. It follows that two of these intersections may be assigned arbitrarily, the cardinal point $\mathscr{A}_{n+2}$ being thereby determined; and a particularly convenient determination arises if two of the free intersections coincide with the absolute points.

We may determine, then, a set of points $F_4, F_5, \ldots, F_{n+2}, \ldots$, such that, if $\mathscr{A}_4$ coincide with $F_4$ two of the free intersections of $w(2) = 0$ and $w(1) = 0$ coincide with the absolute points, and then, $\mathscr{A}_4$ being at $F_4$, and $\mathscr{A}_5$ at $F_5$, two of the free intersections of $w(3) = 0$ and $w(1) = 0$ coincide with the absolute points; and so on, step by step. Further, we may take the points $F_4, F_5, \ldots, F_{n+2}, \ldots$ as new origins of co-ordinates for the several cardinal points, and the various equations, in the parameter $\epsilon$, giving the several sets of free intersections, for varying values of $n$, take a particularly simple form.

Returning to the case $n = 2$, and writing $\epsilon^2+\beta^2 = 0$ in § 9·3, we find that $F_4(X_4, Y_4)$ is given by

$$(\alpha'-\alpha)^3\, X_4+6\beta(\alpha+3\alpha') = 0,$$

$$(\alpha'-\alpha)^3\, Y_4-(\alpha+3\alpha')^2+2\alpha'(\alpha'+\alpha)+12\beta^2 = 0;$$

and then, writing

$$(\alpha'-\alpha)^3(x_4-X_4) = 3\xi, \quad (\alpha'-\alpha)^3\,(y_4-Y_4) = 3\eta,$$

and using $\xi, \eta$ as new co-ordinates for $\mathscr{A}_4$ relative to $F_4$, we find, from §9·3, that the free intersections of $w(2) = 0$ and $w(1) = 0$ are given by the quartic equation

$$\Theta(\epsilon;\ \xi, \eta) \equiv (\epsilon+\alpha)\ (\epsilon+\alpha')\ (\epsilon^2+\beta^2)+\epsilon(\eta\epsilon-\beta\xi) = 0. \tag{15·1}$$

There are then four Burmester points, given by the preceding quartic equation, which we regard as the canonical form of the *Burmester quartic*.

Here $\xi, \eta$ are proportional both to the co-ordinates of the fourth cardinal point $\mathscr{A}_4$, of the first set, relative to the point $F_4(X_4, Y_4)$, and also to the co-ordinates of $\mathscr{A}'_4$ of the second set, relative to the dual point $F'_4(X'_4, Y'_4)$; the axes in each case being parallel to the former axes. For, by the interchange of $\alpha'$ and $\alpha$, we may write, since $\beta' = \beta$,

$$(\alpha-\alpha')^3\,(x'_4-X'_4) = 3\xi, \quad (\alpha-\alpha')^3\,(y'_4-Y'_4) = 3\eta,$$

where
$$(\alpha-\alpha')^3\,X'_4+6\beta(\alpha'+3\alpha) = 0,$$

$$(\alpha'-\alpha)^3\,Y'_4-(\alpha'+3\alpha)^2+2\alpha(\alpha'+\alpha)+12\beta^2 = 0.$$

Thus the equation (15·1) serves to deal both with the Burmester points upon the cubic curve, and also with their duals upon the dual cubic, that is, with the points of the plane $p$, for which the path, relative to $\varpi$, has five-point contact with the corresponding circles of curvature. And the symmetry of the Burmester quartic $\Theta$, in $\alpha$ and $\alpha'$, indicates that these points and their duals correspond, in the sense of §14; indeed, if $\rho_2$ and $\bar{\rho}_2$ be respectively the radii of curvature of the second evolutes of the relative paths, at corresponding points $\epsilon$, upon the dual curves $\Gamma$ and $\Gamma'$, we have

$$(\epsilon-\alpha')^4\,\rho_2+(\epsilon-\alpha)^4\,\bar{\rho}_2 = 0.$$

Thus, if for given $\epsilon$, either of $\rho_2$, $\bar{\rho}_2$ vanish, then the other, in general, vanishes too.

15·2. The first three cardinal points $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ being given, and therefore also the Burmester cubic, the four Burmester points are determined by the fourth cardinal point $\mathscr{A}_4$, and, for varying positions of $\mathscr{A}_4$, these Burmester points constitute a linear series of sets of points, of freedom two, upon the cubic curve.

Further, if one Burmester point $\mathfrak{B}_1(\epsilon_1)$ be assigned, then $\mathscr{A}_4$ lies upon the line $l_1$, given by

$$l_1 \equiv \Theta(\epsilon_1;\ x, y) = 0,$$

referred to $F_4$ as origin; and if, in addition, a second Burmester point $\mathfrak{B}_2(\epsilon_2)$ be assigned, $\mathscr{A}_4$ is determined as the intersection of $l_1$ with a second line $l_2$, given by

$$l_2 \equiv \Theta(\epsilon_2;\ x, y) = 0.$$

Or, corresponding to an assigned position of $\mathscr{A}_4$, there are four lines $l_n$ $(n = 1, 2, 3, 4)$, each passing through $\mathscr{A}_4$, and given by

$$l_n \equiv \Theta(\epsilon_n;\ x, y) = 0;$$

and further, each line $l_n$ is perpendicular to the join of the first cardinal point $\mathscr{A}_1$ and the corresponding Burmester point. If then $\mathscr{A}_4$ be assigned arbitrarily, and if any of the lines $l_n$, passing through $\mathscr{A}_4$, be known, the corresponding Burmester point is thereby determined—as the single remaining intersection, with the cubic $\Gamma$, of the perpendicular from $\mathscr{A}_1$, the node of $\Gamma$.

## 16. Müller's theorem

If one of the Burmester points coincide with the Ball point $\alpha'$, then, from the Burmester quartic

$$(\epsilon+\alpha)\,(\epsilon+\alpha')\,(\epsilon^2+\beta^2)+\epsilon(\eta\epsilon-\beta\xi)=0,$$

it is evident that the parameters of the remaining three points satisfy the relation

$$\epsilon_1\epsilon_2\epsilon_3=\alpha\beta^2;$$

that is, these three Burmester points are collinear. This quite general result appeared, I believe, in the writings of Müller, but, for a particular mechanism, it was given earlier by Tschebycheff.

The result is a special case of a more general property. We have seen that, upon the Burmester cubic, the four Burmester points constitute one set of a linear series of sets of points, of freedom two, corresponding to the arbitrary $\mathscr{A}_4$. If now $\mathscr{A}_4$ be restricted to a line $l_4$, so that one Burmester point $\epsilon_4$ is determined, the remaining three constitute a linear series of freedom one. Each set defines a circle, passing through the three points, and we may show that the aggregate of these circles constitutes a coaxal system, with real points of intersection, if $\epsilon_4$ be real. Indeed, $\epsilon_4$ being assigned arbitrarily, the circle through $\epsilon_1, \epsilon_2, \epsilon_3$ intersects the Burmester cubic again in the point $\epsilon$, given by $\alpha'\epsilon=\alpha\epsilon_4$; that is, in a fixed point.

We may consider the aggregate of conics through three fixed points—the point $\epsilon$ and the circular points—upon the rational cubic curve, and intersecting the curve again in the remaining three Burmester points, which form a linear series of sets, of freedom one. From the quartic $\Theta=0$, such a conic system is a pencil of conics, in this case a system of coaxal circles. If now $\epsilon_4=\alpha'$, the parameter of the Ball point, so that the line $l_4$ coincides with $\mathscr{A}_2B$, then $\epsilon=\alpha$, the real point at infinity upon the Burmester cubic, and the coaxal circles become the line infinity, together with a pencil of straight lines, passing through a fixed point.

Analytically, we may write

$$\psi(E)\equiv(E-e_1)\,(E-e_2)\,(E-e_3),$$

so that $\qquad \sigma\equiv e_1+e_2+e_3=-\alpha-\alpha'-\alpha'\epsilon/\alpha \quad$ and $\quad \sigma_3\equiv e_1e_2e_3=\alpha^2\beta^2/\epsilon,$

and are constants. Then $\sigma_2(\equiv\epsilon_2\epsilon_3+\epsilon_3\epsilon_1+\epsilon_1\epsilon_2)$ may be taken as the single variable of the sets of three points, or of the system of circles. And the equation of the circle through the points $\epsilon_1, \epsilon_2, \epsilon_3$ may be written

$$\beta\psi(\alpha)\,(\epsilon-\alpha)\,(x^2+y^2)+\{(\alpha-\epsilon)\,\sigma_3+\epsilon(\alpha\sigma_2-\sigma_3)\}\,x+\alpha^2\beta\{(\sigma_1-2\alpha+\epsilon)\,y-1\}=0,$$

a linear function of the single variable $\sigma_2$; the circles forming a coaxal system.

If now, as a special case, $\epsilon_4=\alpha'$, the Ball point, so that $\epsilon=\alpha$, the real point at infinity upon the cubic curve, the circles break up into the line infinity, together with

$$(\beta-\sigma_2/\beta)\,x+2(\alpha+\alpha')\,y+1=0,$$

that is, for varying $\sigma_2$, a pencil of straight lines through the point $Q$, upon $\mathscr{A}_1\mathscr{A}_2$, given by

$$2(\alpha+\alpha')\,\mathscr{A}_1Q+1=0. \qquad (16\cdot1)$$

Thus if one Burmester point coincide with the Ball point, the remaining three are collinear, and all lines so arising pass through the fixed point $Q$ upon $\mathscr{A}_1\mathscr{A}_2$. Moreover, we notice that the equation (16·1) is symmetrical in $\alpha$ and $\alpha'$, so that the same point $Q$ arises in the dual configuration; and this point $Q$ plays some part in the subsequent theory.

## 17. Generating curves for the Burmester points

17·1. Through an arbitrary $\mathscr{A}_4$ pass four lines $l_1, l_2, l_3, l_4$, where

$$l(\epsilon) \equiv \Theta(\epsilon;\, x, y) = 0,$$

and we substitute successively the parameters $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, of the four corresponding Burmester points. The origin of co-ordinates is the point $F_4$ of § 15·1, and we may use the same equation for the dual configuration, and origin of co-ordinates being then the dual point $F_4'$.

For varying positions of $\mathscr{A}_4$, these four lines have as envelope a curve of class four; and if the quartic in $\epsilon$ have two coincident roots, $\mathscr{A}_4$ lies upon this envelope, which is given therefore by the vanishing of the discriminant of the quartic equation. This discriminant is of degree six in the coefficients of the quartic, but here the only variable coefficients—which are linear—are those of the terms in $\epsilon$ and $\epsilon^2$; we have therefore, in general, a quintic curve. And this quintic may be regarded as a generating curve for the Burmester points; for the tangents to it from $\mathscr{A}_4$, arbitrarily assigned, give the four corresponding Burmester points, as indicated in § 15·2.

The quintic curve $\mathscr{Q}$ appears as the envelope of the line

$$\epsilon\beta x - \epsilon^2 y - (\epsilon+\alpha)(\epsilon+\alpha')(\epsilon^2+\beta^2) = 0,$$

for varying $\epsilon$; so that if $u$, $v$, $w$ be line co-ordinates, the tangential equation of the curve may be written

$$\mathscr{Q} \equiv (\alpha u - \beta v)(\alpha' u - \beta v)(u^2+v^2) - u^2 v w = 0, \tag{17·1}$$

which is symmetrical in $\alpha$ and $\alpha'$. Or, parametrically, we have

$$\beta\epsilon x + 2(\epsilon^4 - \alpha'\alpha\beta^2) + (\alpha'+\alpha)\,\epsilon(\epsilon^2-\beta^2) = 0,$$
$$\epsilon^2 y + 3\epsilon^4 + 2(\alpha'+\alpha)\,\epsilon^3 + (\alpha'\alpha+\beta^2)\,\epsilon^2 - \alpha'\alpha\beta^2 = 0,$$

the origin of co-ordinates being the point $F_4$ of § 15·1. We have then a rational quintic curve.

The curve touches the line at infinity at two distinct points, the one $\epsilon = 0$, being an ordinary point of contact, and the other $1/\epsilon = 0$, a point of inflexion. The line infinity is then a double bitangent, and this gives the five points of the curve upon that line. Thus, of the four real foci of this curve, of class four, one only is in the finite part of the plane, the remaining three being the points of contact with the line infinity. And, from (17·1), the one real and finite focus is the origin of co-ordinates, the point $F_4$ of § 15·1 or, for the dual quintic, the point $F_4'$.

The cubic which touches the eight non-isotropic tangents, from the four real foci of this curve of class four, reduces, in this case, to a point-pair upon the line infinity, being the two points of intersection, with this line, of the two non-isotropic, and real, tangents $F_4T$, $F_4T'$ to the curve, from the single finite focus $F_4$.

The curve has four cusps, given by

$$3\epsilon^4 + (\alpha'+\alpha)\,\epsilon^3 + \alpha'\alpha\beta^2 = 0,$$

corresponding to the four positions, on the Burmester cubic, at each of which, by suitable choice of $\mathscr{A}_4$, three Burmester points coincide. And there are two double points,

$$x = (\alpha+\alpha')\,(\beta \pm \surd(\alpha\alpha')), \quad y = (\alpha+\alpha')^2/4 - \{\beta \pm \surd(\alpha\alpha')\}^2,$$

the parameters of which are given respectively by

$$2\epsilon^2 + (\alpha+\alpha')\,\epsilon \mp 2\beta\surd(\alpha\alpha') = 0,$$

corresponding to the two cases in each of which, with suitable choice of $\mathscr{A}_4$, there are two pairs of coincident Burmester points.

Further, there is a double bitangent—the line infinity—and one inflexion, also at infinity, the deficiency of the curve being zero, as already apparent.

If $\mathscr{A}_4$ coincides with the origin $F_4$, or $\mathscr{A}'_4$ with $F'_4$, the Burmester quartic reduces to

$$(\epsilon+\alpha)(\epsilon+\alpha')(\epsilon^2+\beta^2) = 0,$$

so that one Burmester point falls at each of the circular points, the remaining two being real points, the one coinciding with the double focus $-\alpha$ on the cubic and the other, $-\alpha'$, with the point on that curve corresponding, in the sense of §14, to the double focus upon the dual cubic.

17·2. The rational quintic $\mathscr{Q}$, like the Burmester cubic $\Gamma$, depends only upon the first three cardinal points $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$, but unlike $\Gamma$, corresponds to one set only of these cardinal points.

Further, we have seen that an arbitrary $\mathscr{A}_4$—$\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$, and therefore the cubic $\Gamma$, and the quintic $\mathscr{Q}$, being given—determines a single set of four Burmester points upon the Burmester cubic; and the correspondence may be regarded as follows. The point $\mathscr{A}_4$ determines four lines $l_1, l_2, l_3, l_4$, tangents from $\mathscr{A}_4$ to $\mathscr{Q}$; and the four Burmester points appear as the four remaining single intersections, with the cubic, of the four perpendiculars from $\mathscr{A}_1$ upon these lines respectively.

Or again; let $D(x, y)$ be the foot of the perpendicular from the origin $F_4$, the single real and finite focus of $\mathscr{Q}$, upon the line $l(\epsilon)$, a tangent to $\mathscr{Q}$. Then we have

$$\epsilon x = \beta(\epsilon+\alpha)(\epsilon+\alpha'), \quad y = -(\epsilon+\alpha)(\epsilon+\alpha'),$$

so that $D$ lies upon a rational cubic curve, the pedal of the quintic $\mathscr{Q}$, the equation of which may be written

$$x^2y+(\alpha x-\beta y)(\alpha' x-\beta y) = 0, \tag{17·2}$$

symmetrical in $\alpha, \alpha'$, and having a node at $F_4$, the normals there being $F_4T$, $F_4T'$, the two real tangents to $\mathscr{Q}$ from $F_4$.

Now, the circle on $F_4\mathscr{A}_4$ as diameter, $\mathscr{A}_4$ being arbitrary, intersects this cubic in two points, coincident at $F_4$, and in four 'free' points $(D)$; the lines $\mathscr{A}_4D$ being the four tangents to $\mathscr{Q}$. Or, the lines $F_4D$ are parallel to the four lines $\mathscr{A}_1\mathfrak{B}_1, \mathscr{A}_1\mathfrak{B}_2, \mathscr{A}_1\mathfrak{B}_3, \mathscr{A}_1\mathfrak{B}_4$, where $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$ are the four Burmester points corresponding to the arbitrary $\mathscr{A}_4$. The rational cubic (17·2) may also be regarded, therefore, as a generating curve for the Burmester points.

17·3. The line $l(\epsilon)$, a tangent to the rational quintic $\mathscr{Q}$, corresponding to the arbitrary point $\epsilon$ upon the Burmester cubic, does not, in general, pass through this point; we may seek the special cases for which the point $\epsilon$ lies upon the line $l(\epsilon)$. And that there are seven such points appear from the following consideration. To an arbitrary point $P(\epsilon)$, upon the cubic $\Gamma$, corresponds one line $l(\epsilon)$, having three intersections $Q_1, Q_2, Q_3$ with $\Gamma$; and to one such point $Q$ there are four lines, through $Q$, tangents to $\mathscr{Q}$, and so four corresponding points $P$, upon $\Gamma$. Hence, between $P$ and $Q$ is a (4, 3) correspondence, and so seven united points, for each of which $P(\epsilon)$ lies upon the line $l(\epsilon)$.

If, in §17·1, we make the appropriate substitutions, we have, as the necessary condition

$$(\epsilon-\alpha')(\epsilon^2+\beta^2)f(\epsilon) = 0,$$

where

$$f(\epsilon) \equiv \epsilon^2\{3(\epsilon+\alpha')^2-4(\alpha'^2+\alpha'\alpha+\alpha^2)\}+3\beta^2(5\epsilon-\alpha)(\epsilon-\alpha).$$

Thus the Ball point $\alpha'$ is one such point; and therefore the line $\mathscr{A}_2B$, that is, the perpendicular from $\mathscr{A}_2$ upon $\mathscr{A}_1\mathscr{A}_3$, is a tangent to the quintic curve $\mathscr{Q}$. The circular points also satisfy the condition, the fourth cardinal point $\mathscr{A}_4$ coinciding then with the single finite focus $F_4$ of $\mathscr{Q}$.

The remaining four points, given by $f(\epsilon) = 0$, are concyclic, since the product of their parameters is $\alpha^2\beta^2$, and they lie upon the circle

$$(\alpha'-\alpha)^2(x^2+y^2)+12\beta x-6(\alpha'+\alpha)y-3 = 0,$$

where, now, $\mathscr{A}_1$ is the origin of co-ordinates. This equation is symmetrical in $\alpha$ and $\alpha'$, so that the same circle arises in the dual configuration. Thus, this circle intersects the dual cubics $\Gamma$ and $\Gamma'$ in twelve points, each of which lies upon the corresponding tangent to $\mathscr{Q}$, or $\mathscr{Q}'$. Further, the centre of the circle, referred to $\mathscr{A}_1$ as origin, is given by the vector sum of $\mathscr{A}_1\mathscr{A}_3$ and $\mathscr{A}_1\mathscr{A}'_3$, and is coincident with the mid-point of $F_4F'_4$.

## 18. Some degenerate cases

Certain special cases of interest arise, for particular values of the constants $\alpha, \alpha'$ and $\beta$. Thus, if $\beta = 0$ the first three cardinal points, of each set, all lie upon the same line; as, for example, in the cardioid displacement of § 6. Or again, if $\alpha+\alpha' = 0$, the two sets of the first three cardinal points are symmetrical about the line through $\mathscr{A}_1$, perpendicular to $\mathscr{A}_1\mathscr{A}_2$.

In the former case, $\beta = 0$, the curve $\mathscr{Q}$ reduces to

$$u^2\{\alpha\alpha'(u^2+v^2)-vw\} = 0,$$

that is, to a coincident point-pair upon the line infinity, in the direction perpendicular to $\mathscr{A}_1\mathscr{A}_2$, together with the parabola

$$x^2 = 4\alpha\alpha'(y+\alpha\alpha'),$$

the focus of which is at $F_4$, and the axis parallel to $\mathscr{A}_1\mathscr{A}_2$. Then the rational cubic curve of § 17·2 reduces to

$$x^2(y+\alpha\alpha') = 0;$$

that is, the line through $F_4$ parallel to $\mathscr{A}_1\mathscr{A}_2$, taken twice, together with the tangent at the vertex of the parabola. We have then the familiar pedal property of the parabola, the focus being the pole.

Also, the Burmester cubic reduces to the line $\mathscr{A}_1\mathscr{A}_2$, together with a circle through $\mathscr{A}_1$, the centre of which is upon $\mathscr{A}_1\mathscr{A}_2$; so that of the four Burmester points two are always coincident, upon $\mathscr{A}_1\mathscr{A}_2$, and the remaining two lie upon the circle, being themselves coincident if $\mathscr{A}_4$ lies upon the parabola.

## 19. The Burmester points as generated by a line and a conic

19·1. The set of Burmester points on the Burmester cubic, corresponding to an assigned position of the fourth cardinal point $\mathscr{A}_4(\xi, \eta)$, is given by the Burmester quartic

$$\Theta(\epsilon;\ \xi,\eta) \equiv (\epsilon+\alpha)(\epsilon+\alpha')(\epsilon^2+\beta^2)+\epsilon(\epsilon\eta-\beta\xi) = 0,$$

and, from § 13, any conic passing through these four points intersects the cubic in two further points $P'(\epsilon')$ and $P''(\epsilon'')$, such that $\epsilon'\epsilon''\alpha' = \alpha\beta^2$; that is, $P'$, $P''$ are collinear with the Ball point $\alpha'$. Or, each set of Burmester points is coresidual with the Ball point.

We write $\epsilon'+\epsilon'' = \beta\lambda/\alpha'$, so that $\lambda$ may be taken as the parameter of the line $P'P''$, passing through the Ball point. Then, corresponding to an assigned value of $\lambda$, we may consider the system of conics passing through $P'$ and $P''$, and intersecting the Burmester cubic in sets of Burmester points. Such a system is a conic net $N(\lambda)$, depending upon $\lambda$, and given by

$$\alpha' N(\lambda;\, \xi, \eta) \equiv \lambda f + f' = 0,$$

where
$$f \equiv \xi f_1 + \eta f_2 + f_3, \quad f' \equiv \xi f'_1 + \eta f'_2 + f'_3,$$
and
$$f_1 = x^2, \quad f_2 = xy, \quad f_3 = -2\beta(\alpha'+\alpha)(x^2+y^2) - \alpha' x - \beta y,$$
$$f'_1 = -\beta(x^2+y^2) + (\alpha'-\alpha)\, xy - x, \quad f'_2 = \alpha x^2 + \beta y^2,$$
$$f'_3 = 2(\alpha'+\alpha)(\alpha\alpha'+\beta^2)(x^2+y^2) + \beta(3\alpha'+\alpha)\, x + \alpha'(\alpha'+3\alpha)\, y + \alpha'.$$

For assigned $\mathscr{A}_4(\xi, \eta)$ and varying $\lambda$, the system is a conic pencil, and, in particular, if $\xi = 0 = \eta$, a system of coaxal circles.

Each pair of conics of the net $N(\lambda)$, corresponding to an assigned value of $\lambda$, has the line $P'P''$ as a common chord, the remaining common chords, for all pairs of conics, having a point in common, the point $Q$, of §16, upon $\mathscr{A}_1\mathscr{A}_2$, where $2(\alpha'+\alpha)\,\mathscr{A}_1 Q + 1 = 0$. This point $Q$, therefore, is independent of the $\lambda$-line chosen, and also plays the same role in the dual configuration, its co-ordinates being symmetrical in $\alpha$ and $\alpha'$.

With regard to $P'$ and $P''$ as absolute points, the conics of the net $N$ are 'circles' of which $Q$ is the radical centre.

In general, there is a cubic curve $U(\lambda)$ of which $N(\lambda)$ is the polar net. The sets of Burmester points may be regarded, therefore, as generated by this cubic $U(\lambda)$, being given by the residual intersections with the Burmester cubic, other than $P'$ and $P''$, of the polar net of $U(\lambda)$. Also, $U(\lambda)$ depends only upon $\lambda$ and the first three cardinal points.

Further, the cubic $U(\lambda)$ consists of the line $P'P''$, together with a conic touching $QP'$ and $QP''$ at $P'$ and $P''$ respectively. Similarly, the Jacobian $J(\lambda)$, of the net $N(\lambda)$, that is, the Hessian of the cubic $U(\lambda)$, consists of the line $P'P''$, together with a second conic, touching $QP'$ and $QP''$ at $P'$ and $P''$ respectively.

To an assigned $\mathscr{A}_4(\xi, \eta)$ corresponds one set of Burmester points, and so one conic of the net $N(\lambda)$, $\lambda$ being assigned; and there is one point $X, Y$, the pole of this conic with respect to the cubic $U(\lambda)$, and conversely. Thus the pole $X, Y$ is in $(1, 1)$ correspondence with the point $\mathscr{A}_4(\xi, \eta)$, the correspondence depending upon the arbitrary parameter $\lambda$. So that we have a collineation between $\xi, \eta$ and $X, Y$ given by $X(\lambda;\, \xi, \eta)$, $Y(\lambda;\, \xi, \eta)$.

We may gather together the preceding results as follows: to an arbitrarily assigned $\mathscr{A}_4(\xi, \eta)$ corresponds one point $X, Y$, the polar conic of which, with respect to the cubic $U(\lambda)$, gives the corresponding set of four Burmester points by its residual intersections, other than $P'$ and $P''$, with the Burmester cubic.

19·2. The arbitrary parameter $\lambda$ may be chosen so that the $\lambda$-line passes also through the dual Ball point, and then this line plays identical roles in the dual configurations; but it is not possible for the $\lambda$-conic to be chosen similarly. For this would imply two intersections of the Burmester cubic with its dual, elsewhere than at $\mathscr{A}_1$ and the absolute points, and there are no such intersections.

In the collineation between the point $X, Y$ and the fourth cardinal point $\mathscr{A}_4(\xi, \eta)$, the $\lambda$-line, $L$, and the line $B\mathscr{A}_2$ correspond, where $B$ is the Ball point; and it is convenient to write $\lambda = 0$, so that these two lines coincide. The corresponding conic is

$$S \equiv \{4\beta(\alpha'-\alpha)\, x + \alpha'(\gamma+2\gamma')\, y + 4\alpha'\}^2 + 3\alpha'\gamma^2(4\alpha x^2 + \alpha' y^2) = 0,$$

where
$$\gamma = \alpha + 3\alpha' \quad \text{and} \quad \gamma' = 3\alpha + \alpha'.$$



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Then each conic of the polar net of the cubic

$$U \equiv LS = 0$$

intersects the Burmester cubic in two fixed points, upon the line $L$, and in four residual points, forming a set of Burmester points.

Further, if we write

$$\omega_1 \equiv \gamma(\xi - \beta\gamma'),$$
$$\omega_2 \equiv 8\beta^2(\alpha' + \alpha) - \alpha'(\alpha' - \alpha)^2 - 4(\beta\xi - \alpha'\eta),$$
$$\omega_3 \equiv 2\beta(\alpha' - \alpha)\,\omega_1 + \alpha\gamma'\omega_2 - \alpha\alpha'\gamma^3,$$

the set of Burmester points, corresponding to $\mathscr{A}_4(\xi, \eta)$, is given by the polar conic of $X, Y$ with respect to the cubic $U$, where

$$\alpha X/\omega_1 = \alpha' Y/\omega_2 = 2\alpha\alpha'/\omega_3.$$

And the line $B\mathscr{A}_2$, given by $\omega_2 + \alpha'\gamma^2 = 0,$

is one of the three self-corresponding lines in the collineation between $\xi, \eta$ and $X, Y$.

## 20. Certain special nets of conics

For certain values of $\lambda$ the preceding investigation breaks down; because, for such values, the corresponding net $N(\lambda)$ contains at least one double line, and there is not then any generating cubic $U(\lambda)$. These cases are of some interest, for they correspond to certain special configurations of the Burmester points; indeed, the four points are then collinear, and, since they lie upon a cubic, there must be coincidences amongst them. And then, also, the fourth cardinal point $\mathscr{A}_4$ lies upon the quintic curve $\mathscr{Q}$. We proceed to examine these special cases here.

We may write, temporarily, $\lambda/\mu$ in place of $\lambda$, so that the conic net is

$$\lambda f + \mu f' = 0;$$

and this net contains a double line if the ratio $\lambda/\mu$ have any of the four values given by the following, namely,

$$\mu = 0,$$
$$\lambda/\mu + \alpha'(3\alpha' + \alpha)/\beta = 0,$$
$$(\lambda/\mu)^2 - 4\alpha\alpha' = 0.$$

(1) If $\mu = 0$, the corresponding $\lambda$-line passes through the first cardinal point $\mathscr{A}_1$, the node of the Burmester cubic, and $P', P''$ coincide there, with parameters $0, \infty$; the double line is the line $\mathscr{A}_1\mathscr{A}_2$ taken twice, and the four Burmester points coincide at $\mathscr{A}_1$, the parameters being $0, \infty, \infty, \infty$. The fourth cardinal point $\mathscr{A}_4$ is then the point of inflexion, on the line infinity, of the quintic curve $\mathscr{Q}$. And the Jacobian of the net, and of the cubic, reduces to three lines, the line $\mathscr{A}_1 B$, $B$ being the Ball point, and the line $\mathscr{A}_1\mathscr{A}_2$ taken twice.

(2) If $\lambda/\mu + \alpha'(3\alpha' + \alpha)/\beta = 0$, the $\lambda$-line passes through the point $Q$ of §16, upon $\mathscr{A}_1\mathscr{A}_2$, and then, although $P', P''$ are not coincident, the lines $QP'$, and $QP''$ coincide. The double line is this line $BQ$, taken twice, and of the four Burmester points, two coincide at $B$, and the other two lie one each at the remaining intersections of $BQ$ with the Burmester cubic. The fourth cardinal point $\mathscr{A}_4$ is the point of contact of the tangent $\mathscr{A}_2 B$ to the quintic $\mathscr{Q}$; and

the cubic $U$ is the line $BQ$, taken three times, while the Jacobian of the net is this line $BQ$ taken twice, together with the line through $\mathscr{A}_1$

$$(3\alpha'+\alpha)\,x-2\beta y=0.$$

(3) If $(\lambda/\mu)^2-4\alpha\alpha'=0$, the $\lambda$-line coincides with one or other of the two tangents $BT_1$, $BT_2$, to the Burmester cubic, from the Ball point upon the curve, so that $P', P''$ coincide with $T_1$ or $T_2$ of parameter $\epsilon'=\epsilon''=\pm\beta\surd(\alpha/\alpha')$.

We take $T_1$ as corresponding to $\lambda/\mu=+2\surd(\alpha\alpha')$, and then the double line $l_1$ of the net passes through $T_1$, and is the harmonic conjugate of $T_1\mathscr{A}_1$ with respect to the line-pair $T_1B, T_1Q$. The cardinal point $\mathscr{A}_4$ is at one of the two nodes of the quintic $\mathscr{Q}$, and the four Burmester points coincide in pairs at the two intersections of $l_1$, other than $T_1$, with the Burmester cubic. The cubic $U$ reduces to three lines through $T_1$, of which $l_1$ is one; and the Jacobian of the net is the three lines $T_1B$, $T_1\mathscr{A}_1$ and $l_1$.

There is a second case, for the net contains also the line $\mathscr{A}_1T_1$ taken twice, the cardinal point $\mathscr{A}_4$ being then the point of ordinary contact of the curve $\mathscr{Q}$ with the line infinity. Here, the four Burmester points coincide with $\mathscr{A}_1$, having parameters $0, 0, \infty, \infty$.

For the value $\lambda/\mu=-2\surd(\alpha\alpha')$, we have similar configurations involving the point of contact $T_2$.

In each of the preceding cases, in which the conic net $N$ contains at least one double line, the four Burmester points may be collinear; but also they may be collinear, because of coincidences, when the net does not contain any double line. For, if the fourth cardinal point $\mathscr{A}_4$ fall at one of the four cusps of the quintic curve $\mathscr{Q}$, three of the corresponding Burmester points coincide, so that there are four further cases, in each of which the Burmester points are collinear.

## 21. Some general results

Let one Burmester point, $\epsilon_4$, be fixed arbitrarily, so that the fourth cardinal point $\mathscr{A}_4$ lies upon a fixed line $l_4$; then the conic net $N(\lambda)$, for assigned $\lambda$, reduces to a conic pencil $\varpi$, three of the base points of which lie upon the Burmester cubic. A conic of this pencil has three 'free' intersections with the cubic—the three remaining Burmester points—which constitute then a linear series of sets, each of three points, of freedom one. Such a set of three points, together with two arbitrary and fixed points $K(\epsilon_k)$, $L(\epsilon_l)$ upon the curve, define a conic, intersecting the cubic in a sixth point $\epsilon'(\equiv\alpha\beta^2\epsilon_4/\alpha'\epsilon_k\epsilon_l)$, which is fixed. The aggregate of such curves is, in general, a quadratic $\infty^1$ of conics, but since, here, the sixth intersection $\epsilon'$ is fixed, we have a conic pencil $\varpi'$. The three remaining Burmester points appear then as the free intersections of this pencil $\varpi'$ with the Burmester cubic, three of the base points of the pencil being upon the curve.

In particular, if $K$ and $L$ coincide with the circular points, upon the cubic, the pencil $\varpi'$ is a system of coaxal circles, as in §16.

As a further specialization, let the arbitrarily assigned Burmester point, $\epsilon_4$, coincide with the Ball point $\alpha'$; then the third base point of the pencil $\varpi'$ is given by $\epsilon'=\alpha$, that is, it coincides with the real point at infinity upon the Burmester cubic. The system of coaxal circles breaks up then into the line infinity, containing the points of parameters $\epsilon_k, \epsilon_l$ and $\alpha$, together with a pencil of lines passing through the fourth base point of the pencil $\varpi'$, which, in this case, is the point $Q$ upon $\mathscr{A}_1\mathscr{A}_2$, where $2(\alpha'+\alpha)\mathscr{A}_1Q+1=0$. Thus the three free Burmester points are collinear, and we have the general result, due to Müller, of § 16, the special case of which, for the three-bar mechanism, was given earlier by Tschebycheff.

If the line of the pencil, through $Q$, pass also through the Ball point, the four Burmester points are collinear, two being at the Ball point.

The result due to Müller follows also for general positions of $K(\epsilon_k)$, $L(\epsilon_l)$; for then, if $\epsilon_4=\alpha'$, the sixth intersection of the pencil $\varpi'$, with the Burmester cubic, is given by $\epsilon_k\epsilon_l\epsilon'=\alpha\beta^2$; that is, this intersection is collinear with $K$, $L$. Thus again, if one Burmester point coincide with the Ball point, the remaining three Burmester points are collinear, and all lines so arising pass through a fixed point, the fourth base point $Q$ of the pencil $\varpi'$.

## 22. The $\rho_3$ points

22·1. The position of the fifth cardinal point $\mathscr{A}_5$ determines the aggregate of points $P_\varpi$ for which the radius of curvature $\rho_3$ of the third evolute of the corresponding relative path vanishes, and, from § 9·2, such points lie upon $w(3)=0$, a curve of degree eleven. This curve has only four 'free' intersections with the Burmester cubic, the remaining 29 fixed intersections being at the first cardinal point $\mathscr{A}_1$ (17), at the Ball point (2), and at the absolute points (10). Thus, upon the cubic, we have a linear series of sets, each of four points, of freedom two.

If $\mathscr{A}_4$ be given, two such points may be assigned arbitrarily upon the cubic, $\mathscr{A}_5$ being thereby determined, and, in particular, $\mathscr{A}_4$ being at $F_4(X_4, Y_4)$, these coincide with the absolute points provided that $\mathscr{A}_5$ be at $F_5(X_5, Y_5)$, where

$$(\alpha'-\alpha)^4X_5-60\beta^3+15\beta(\alpha'+\alpha)(5\alpha'+\alpha)=0,$$
$$(\alpha'-\alpha)^4Y_5+60\beta^2(\alpha+2\alpha')-15\alpha'(\alpha'+\alpha)^2=0.$$

It is convenient to take $F_5$ as origin of co-ordinates $(\xi',\eta')$, for $\mathscr{A}_5$, and to write

$$(\alpha'-\alpha)^4(x_5-X_5)=15\xi',\quad (\alpha'-\alpha)^4(y_5-Y_5)=15\eta'.$$

Then, from § 9·3, the point $\epsilon$ upon the Burmester cubic is a $\rho_3$ point if

$$\Theta_3\equiv 2\alpha\beta U-\epsilon^2(\epsilon\xi'+\beta\eta')+\epsilon\{\epsilon(\alpha'-\epsilon)(\epsilon\xi+\beta\eta)+2\beta(\epsilon-\alpha)(\beta\xi-\epsilon\eta)\}=0,$$

where $$U\equiv(\epsilon+\alpha)(\epsilon+\alpha')(\epsilon^2+\beta^2).$$

We refer to the equation $$\Theta_3(\epsilon)=0$$

as the $\rho_3$ quartic; clearly, there is a dual quartic

$$\Theta_3'(\epsilon)=0,$$

obtained by the interchange of $\alpha$ and $\alpha'$. And, unlike the Burmester quartic and its dual, these two quartics, being unsymmetrical in $\alpha$ and $\alpha'$, are not identical.

22·2. There are four points upon the Burmester cubic, given by

$$(\alpha'-\alpha)\epsilon^4+4\alpha'\alpha\epsilon^3+2\alpha'\alpha^2\beta^2=0,$$

at each of which, by proper choice of $\mathscr{A}_4$ and $\mathscr{A}_5$, the four $\rho_3$ points coincide; then, $\epsilon$ being one of the roots of this equation, $\xi=2\alpha\beta(1-\alpha\alpha'\beta^2/\epsilon^4)$,

so that $\mathscr{A}_4$ lies upon one or other of four lines parallel to $\mathscr{A}_1\mathscr{A}_2$, and $\mathscr{A}_5$ lies upon a corresponding line, the four lines so arising being parallel to

$$(\alpha'+2\alpha)x+3\beta y=0.$$

Also, if one $\rho_3$ point be given by

$$\epsilon(1-\xi/2\alpha\beta) = \alpha',$$

the remaining three are collinear, and all lines so arising pass through a point $Q_3$, where $\mathscr{A}_1 Q_3$ is perpendicular to $\mathscr{A}_1\mathscr{A}_2$, and

$$\beta\{3\alpha\alpha'+\epsilon(2\alpha'-\alpha)\}\mathscr{A}_1 Q_3+\alpha'\epsilon = 0.$$

In particular, if $\epsilon = \alpha'$, that is, the Ball point, then

$$2(\alpha+\alpha')\beta\mathscr{A}_1 Q_3+\alpha' = 0;$$

we have, therefore, a property somewhat similar to that given by Müller for the Burmester points; but here, owing to the absence of symmetry in $\alpha$ and $\alpha'$, the point $Q_3$ does not coincide with its dual.

More generally, if one $\rho_3$ point be assigned arbitrarily, the remaining three lie upon one of a system of coaxal circles.

And the four points are concyclic if $\mathscr{A}_4$ lie upon the line

$$\xi = 2\beta(\alpha-\alpha').$$

Further, we may regard $\mathscr{A}_4(\xi,\eta)$ as assigned; then through an arbitrary $\mathscr{A}_5(\xi',\eta')$ pass four lines, given by

$$\Theta_3(\xi',\eta') = 0,$$

enveloping a curve of class four, which may be regarded as a generating curve for the $\rho_3$ points. In particular, if $\xi = 0 = \eta$, this curve is given tangentially by

$$2\alpha(\beta u+\alpha v)\,(\beta u+\alpha' v)\,(u^2+v^2)+u^2vw = 0.$$

Again, the four $\rho_3$ points are coresidual with the point $\epsilon'$, upon the Burmester cubic, where

$$\epsilon'(1-\xi/\alpha\beta) = \alpha',$$

that is, with the Ball point, if $\xi = 0$. Then, as in § 19·1, we have a system of generating cubics $U_3(\lambda)$, such that the polar net of $U_3(\lambda)$ intersects the cubic in two fixed points, collinear with the point $\epsilon'$, and in four 'free' intersections, which are sets of $\rho_3$ points.

22·3. The interest attaching to the $\rho_3$ points centres, perhaps, in the possibility of their coincidences with the Burmester points, as in three-bar displacement; for any such coincidence implies that the corresponding relative path has six-point contact with its circle of curvature. Here we may regard $\mathscr{A}_4(\xi,\eta)$ as arbitrary; then two of the $\rho_3$ points coincide with the absolute points provided that $\mathscr{A}_5(\xi',\eta')$ be given by

$$\xi'_0 = (2\alpha+\alpha')\,\xi-3\beta\eta, \quad \eta'_0 = 3\beta\xi+(2\alpha+\alpha')\,\eta.$$

We regard this as the standard position of $\mathscr{A}_5$, and we write

$$\xi' = \xi'_0+\xi'', \quad \eta' = \eta'_0+\eta'';$$

then the $\rho_3$ quartic takes the simpler form

$$\Theta_3(\epsilon;\,\xi'',\eta'')\equiv 2\alpha\beta U-\epsilon(\epsilon+2\alpha)\,(\epsilon^2+\beta^2)\,\xi-\epsilon^2(\epsilon\xi''+\beta\eta'') = 0.$$

The two quartics $\Theta = 0$, $\Theta_3 = 0$ have four roots in common if

$$\xi = 0, \quad \xi'' = 0, \quad \eta''+2\alpha\eta = 0,$$

that is, if $\mathscr{A}_4$ lies upon the line through $F_4$ parallel to $\mathscr{A}_1\mathscr{A}_2$, and $\mathscr{A}_5$ lies upon the line through $F_5$ parallel to

$$a'x+3\beta y = 0.$$

Clearly, there are $\infty^1$ such cases.

Further, it is the case that if the two quartics have three roots in common, other than $\epsilon = 0$, the fourth root also is common. For, $\Theta = 0$ being the Burmester quartic, we have

$$2\alpha\beta\Theta - \Theta_3 = \epsilon^2 Q,$$

where $Q$ is quadratic in the parameter $\epsilon$, given by

$$Q \equiv \xi\epsilon^2 + (2\alpha\xi + \xi'')\,\epsilon + \beta(2\alpha\eta + \beta\xi + \eta'').$$

And then $$\xi = 0, \quad \xi'' = 0, \quad \eta'' + 2\alpha\eta = 0.$$

In the case of importance, that of three-bar displacement, two of the $\rho_3$ points coincide respectively with two Burmester points, upon the Burmester cubic; and, if these be given by the parameters $\epsilon_1, \epsilon_2$, we have

$$1/\xi = -(\epsilon_1 + \epsilon_2)/(2\alpha\xi + \xi'') = \epsilon_1\epsilon_2/\beta(2\alpha\eta + \beta\xi + \eta''),$$

so that $$\xi'' + \xi(2\alpha + \epsilon_1 + \epsilon_2) = 0 \quad \text{and} \quad \beta\eta'' = (\epsilon_1\epsilon_2 - \beta^2)\,\xi - 2\alpha\beta\eta.$$

It follows that the fifth cardinal point $\mathscr{A}_5$ coincides with one or other of six points, which lie by threes upon four straight lines.

## References

Of the many references which may be given to investigations in the realm of plane kinematics, a few only have been indicated, as being associated, more or less closely, with the matters concerned in the foregoing paper.

Studies of the very important three-bar mechanism have prompted much in the general theory of plane kinematics, and have given rise to an extensive literature, over a long period. Amongst many others, we may mention the names of Roberts, Cayley and Darboux. Perhaps the most comprehensive treatment of the theory of this mechanism, to the present time, dealing with the three-bar sextic curve, is that given by Bennett (1922); we may also refer to a paper by Morley (1923).

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